N/D method to finite-volume analysis of H-dibaryon including left-hand cut

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Cake Seminar
Newport News, VA • 2026/05/13

Based on: A.Rodas, L.Qiu and JPAC members, arXiv: 2605.22957
L.Qiu, C.Gong and Q.Zhao, PhysRevD.109.076016

Outline

- Role of left-hand cut phenomenologically and in finite volume

  left-hand-cut issues, effective-range expansions and N/D representation, NN scatterings

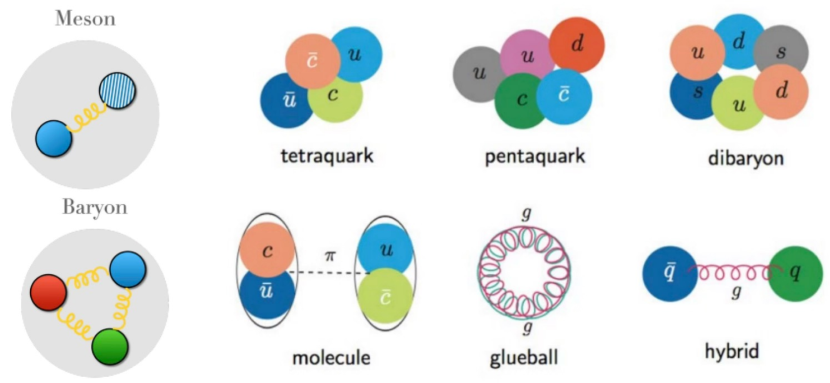
- N/D method to finite-volume analysis of H-dibaryon including left-hand cut

  status of H-dibaryon, Lüscher's and N/D quantization conditions, FV analysis of H-dibaryon

Role of left-hand cut phenomenologically and in finite volume

Status of hadron spectroscopy

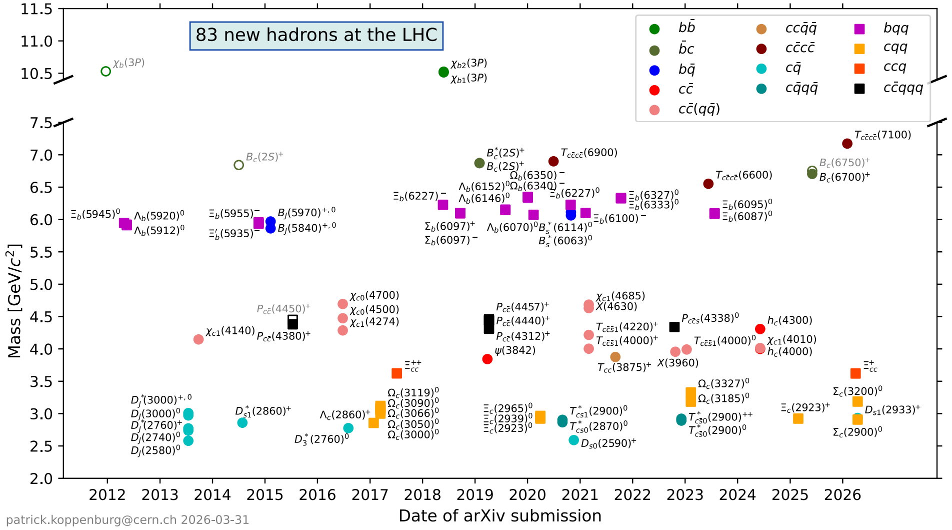
From LiuPan An's talk



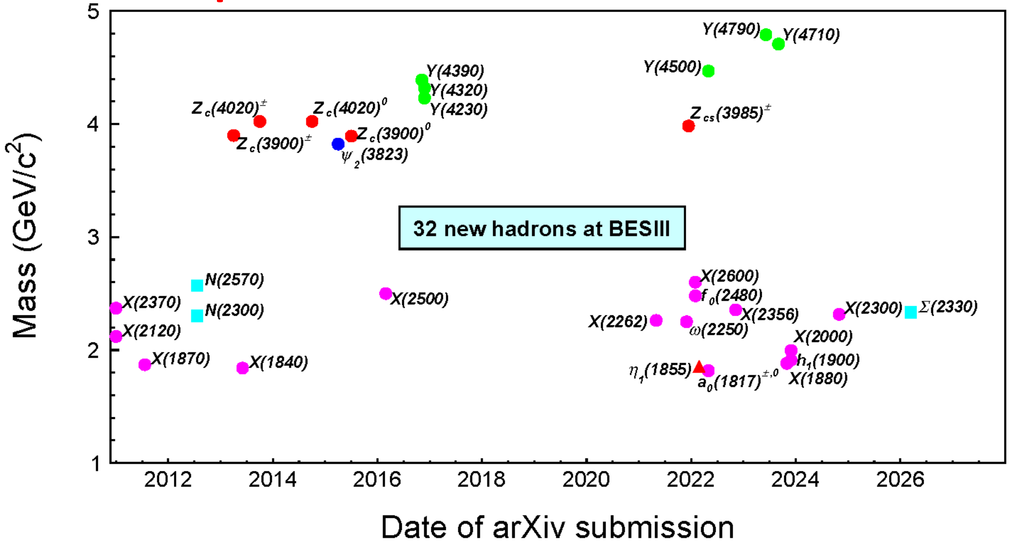
- From 2011 to now,
  - 83 new hadrons discovered at the LHC
  - 32 new hadrons discovered at BESIII
- Most of their masses are associated with two-hadron thresholds
  - Dominant molecular nature
  - Mediated by light-meson exchange: $\pi, K, \eta, \omega, \rho, \phi$
- Hard scale $\Lambda \sim \Lambda_\chi \approx$ masses of vector mesons
- Soft scale $Q \sim m_\pi$
  - ☞ where things get complicated multi-hadron

What is the role of the pion?

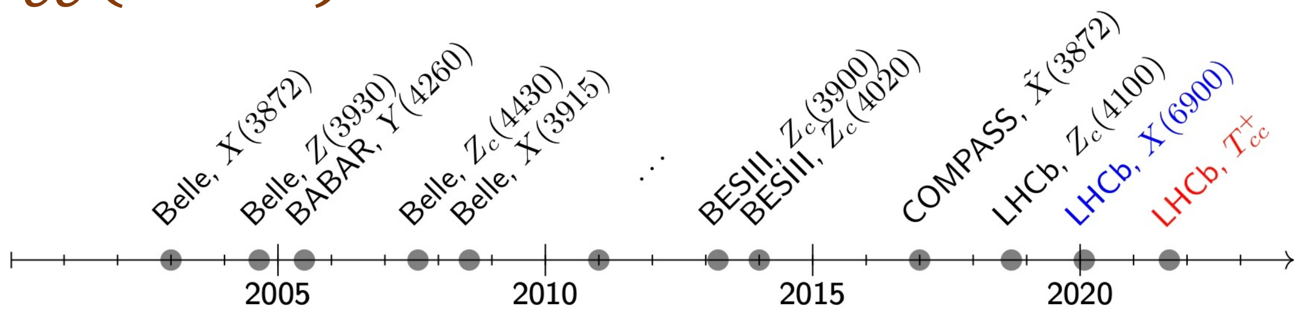
https://www.koppenburg.ch/particles.html



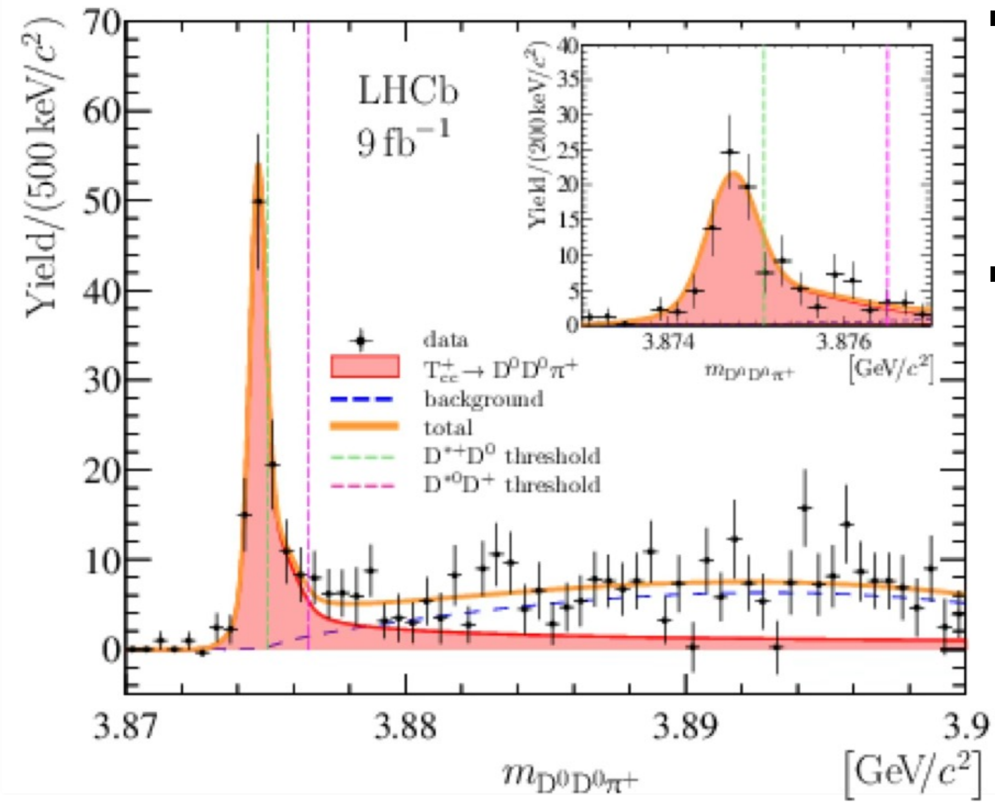
https://english.ihep.cas.cn/bes/re/pu/NewParticles/



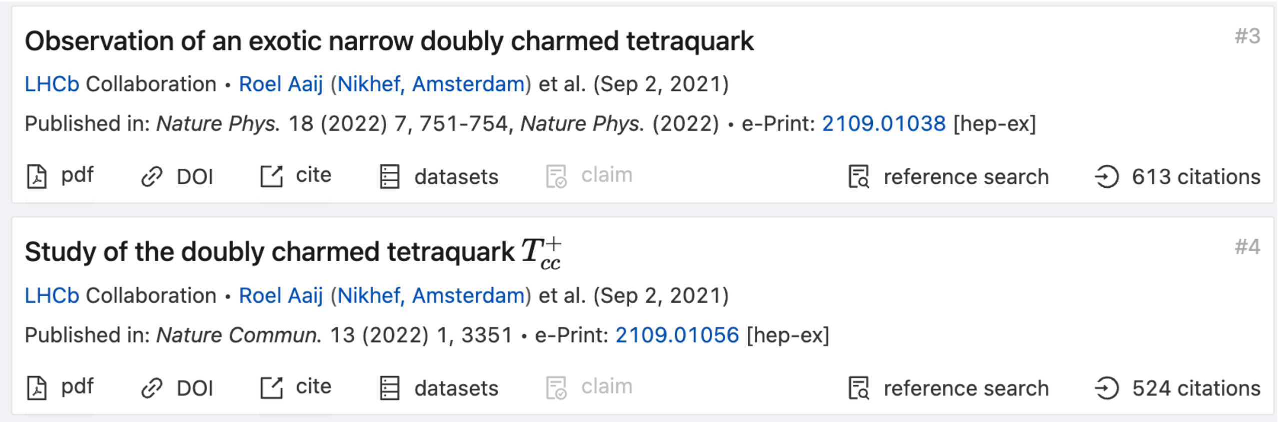
Closer look: $T_{cc}^{+}(3875)$



LHCb,NP18(2022)7, NC13(2022)13351

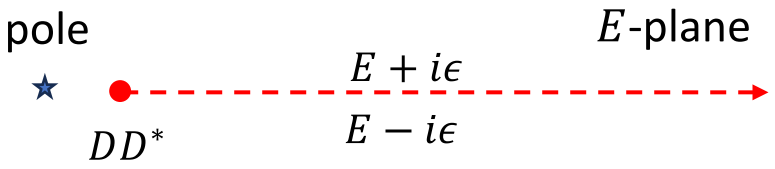


- First double-charm tetraquark ($cc\bar{u}\bar{d}$) observed in $D^0D^0\pi^+$ by LHCb
- Generally regarded as $IJ^P = 01^+$ DD^* molecule
  - BW fit: $\delta m_{\mathrm{BW}} = -273 \pm 61\mathrm{keV}$, $\Gamma_{\mathrm{BW}} = 410 \pm 165\mathrm{keV}$ $\Gamma_{D^*} = 60\mathrm{keV}$
  - Unitarized fit: $\delta m_{\mathrm{pole}} = -360\mathrm{keV}$, $\Gamma_{\mathrm{pole}} = 48\mathrm{keV}$
  - ☞ Dynamics, especially π, is important for precise results
- It INSPIREd a lot of studies pheonomenologically and on LQCD


Observation of an exotic narrow doubly charmed tetraquark #3
LHCb Collaboration • Roel Aaij (Nikhef, Amsterdam) et al. (Sep 2, 2021)
Published in: *Nature Phys.* 18 (2022) 7, 751-754, *Nature Phys.* (2022) • e-Print: 2109.01038 [hep-ex]
pdf DOI cite datasets claim reference search 613 citations

Study of the doubly charmed tetraquark T_{cc}^{+} #4
LHCb Collaboration • Roel Aaij (Nikhef, Amsterdam) et al. (Sep 2, 2021)
Published in: *Nature Commun.* 13 (2022) 1, 3351 • e-Print: 2109.01056 [hep-ex]
pdf DOI cite datasets claim reference search 524 citations

$DD^*(\bar{D}^*)$ scattering in EFT



In the naïve dimensional analysis or power counting,

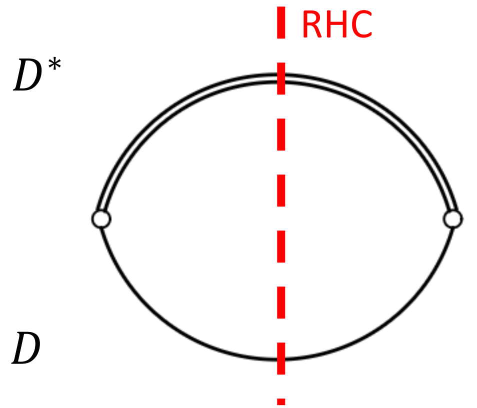
$$\frac{g^2 M_{DD^*}\mu}{4\pi f_\pi^2} \approx \frac{1}{20} \sim \frac{1}{10} \quad \ll \quad \frac{g_A^2 M_N m_\pi}{8\pi f_\pi^2} \approx \frac{1}{2}$$

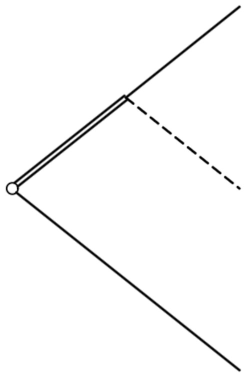
S.Fleming et al.,PRD76.034006
Jansen et al., PRD.89.014033
Lin Dai et al., PRD.101.054024
M.P.Valderrama, PRD.85.114037

wavefunction renormalization

decay

LO: $\mathcal{O}(Q^{-1})$





Effective range expansion (ERE):

$$T(E) = -\frac{\mu_{DD^*}}{2\pi}\frac{1}{\mathrm{p}\cot\delta \mp ip}$$

$$p\cot\delta = \frac{1}{a} + \frac{1}{2}rp^2 + \mathcal{O}(p^4)$$

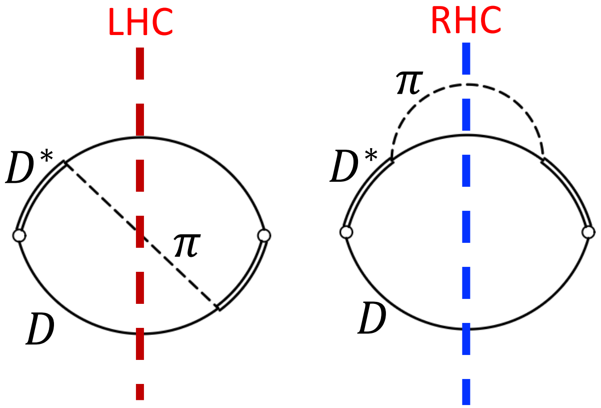
☞ Effective up to hundreds of MeV in p

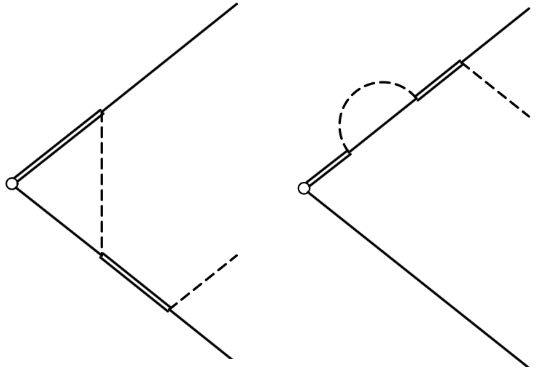
Pion is perturbative:

☞ $0.98\left(m_\pi^{phy}\right)^2 \lesssim m_\pi^2 \lesssim 2\left(m_\pi^{phy}\right)^2$

☞ $DD - DD^* - D^*D^*$ coupling

NLO: $\mathcal{O}(Q^0)$





The π-exchange and self-energy $D^* \to D\pi$ introduce additional singularities to the above $T(E)$ or $p\cot\delta$

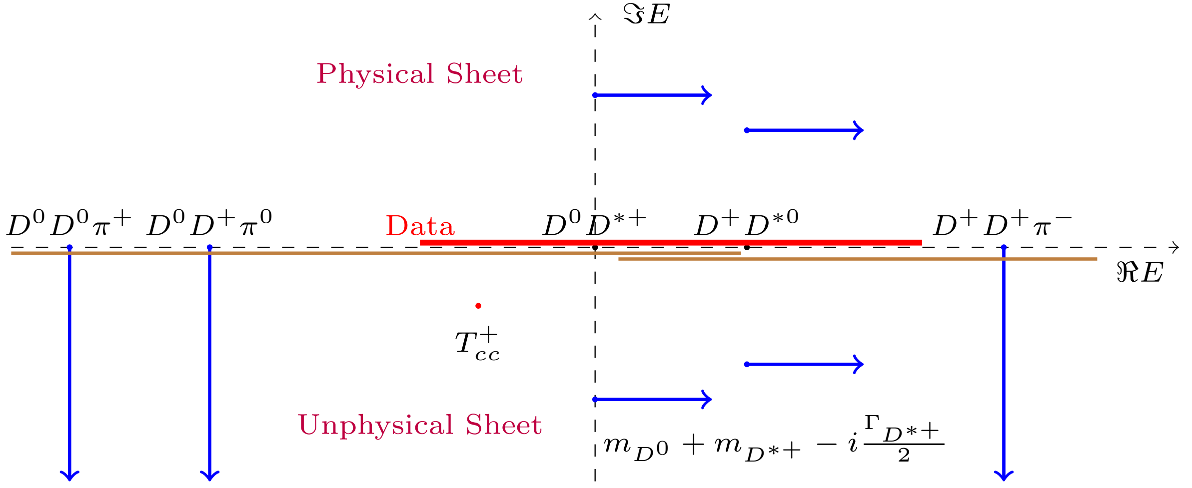
☞ Much smaller convergence radius of ERE

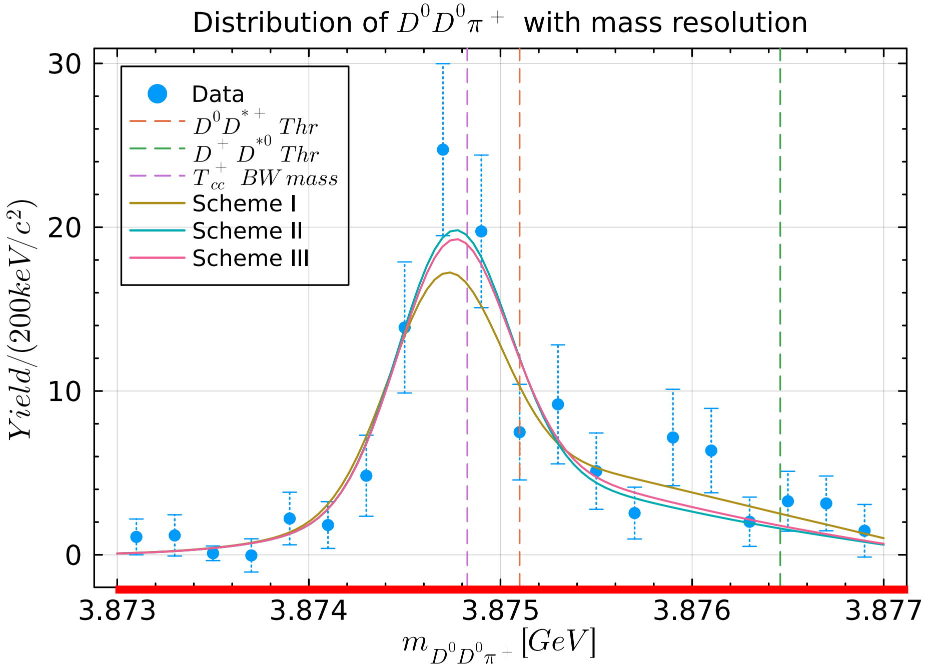
Unitarization of $D^0D^{*+} - D^+D^{*0}$ scattering

M.L.Du et al., PRD.105.014024
L.Qiu et al., PRD.109.076016
X.Zhang, PRD.109.094010

$$T_{\alpha\beta}(p,k;E) = V_{\alpha\beta}(p,k;E) + \sum_\gamma \int \frac{d^3q}{(2\pi)^3} \frac{V_{\alpha\gamma}(p,q;E)T_{\gamma\beta}(q,k;E)}{E - m_1 - m_2 - \frac{q^2}{2\mu_\gamma} + i\frac{\Gamma_\gamma(q;E)}{2}}$$

$$V = V_{\eta,\rho,\omega,\phi} + V_\pi$$



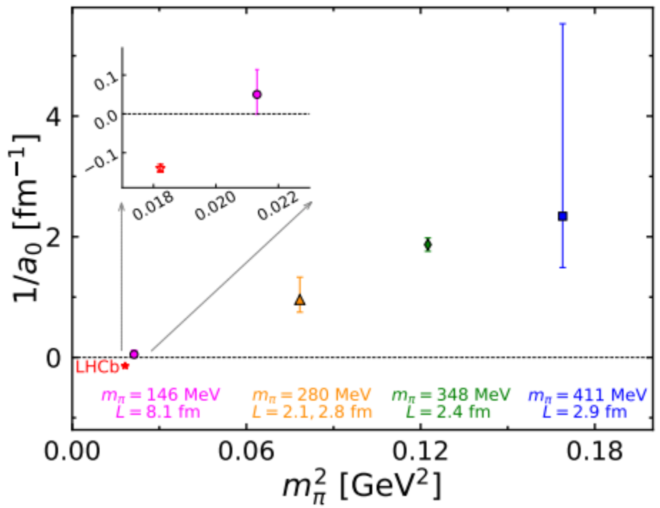


Even perturbatively, the treatment on π is important for
- ☞ Near-threshold dynamics
- ☞ Pole position

Especially relevant on Lattice study
when m_π is slightly larger!

Schemes	Pole of T_{cc}^+[keV]
Scheme I: Breit-Wigner Γ_{D^*}	$-379.9^{+15.7}_{-15.5} - i37.0^{+0.0}_{-0.0}$(RS-I)
Scheme II: Dynamical $\Gamma_{D^*}(E,q)$	$-347.4^{+11.5}_{-11.3} - i18.7^{+0.2}_{-0.2}$(RS-II)
Scheme III: $\Gamma_{D^*}(E,q)$ w/ π-exchange	$-350.4^{+18.3}_{-18.2} - i24.6^{+0.3}_{-0.2}$(RS-II)

Lattice study on T_{cc}^{+} and LHC problem

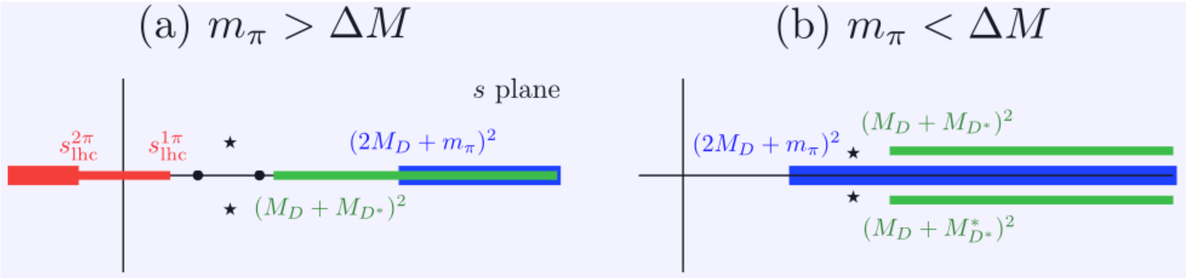


Y. Ikeda et al. (HAL QCD), 1311.6214;
P.Junnarkar et al., PRD99,034507
M. Padmanath and S. Prelovsek, 2202.10110;
S. Chen et al., 2206.06185;
Y. Lyu et al. (HAL QCD), 2302.04505;
T.Whyte et al.(HadSpec),PRD111,034511

M.-L. Du et al., PRL 131 (2023) 131903

(b) For physical isospin-average masses,

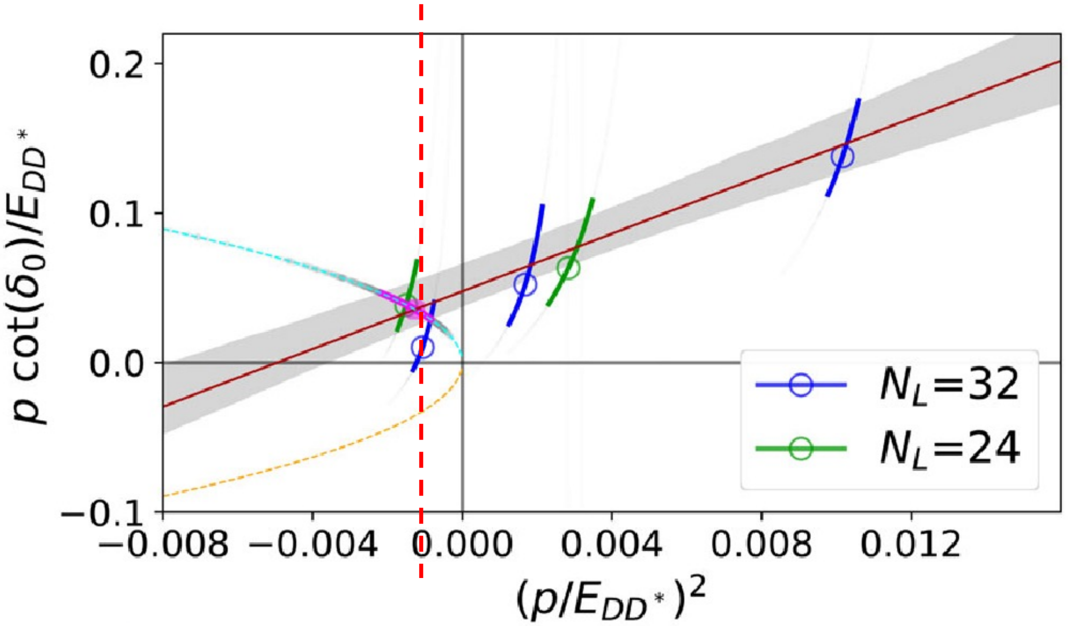
$$\left(\frac{p_{\mathrm{rhc}_3}}{E_{DD^*}}\right)^2 = -0.0004, \qquad \left(\frac{p_{\mathrm{lhc}}^{\mathrm{OPE}}}{E_{DD^*}}\right)^2 > 0$$



(a) For $m_\pi = 280\mathrm{MeV}$, $M_D = 1927\mathrm{MeV}$, $M_{D^*} = 2049\mathrm{MeV}$,

$$\left(\frac{p_{\mathrm{rhc}_3}}{E_{DD^*}}\right)^2 = 0.02, \qquad \left(\frac{p_{\mathrm{lhc}}^{1\pi}}{E_{DD^*}}\right)^2 = -0.001$$

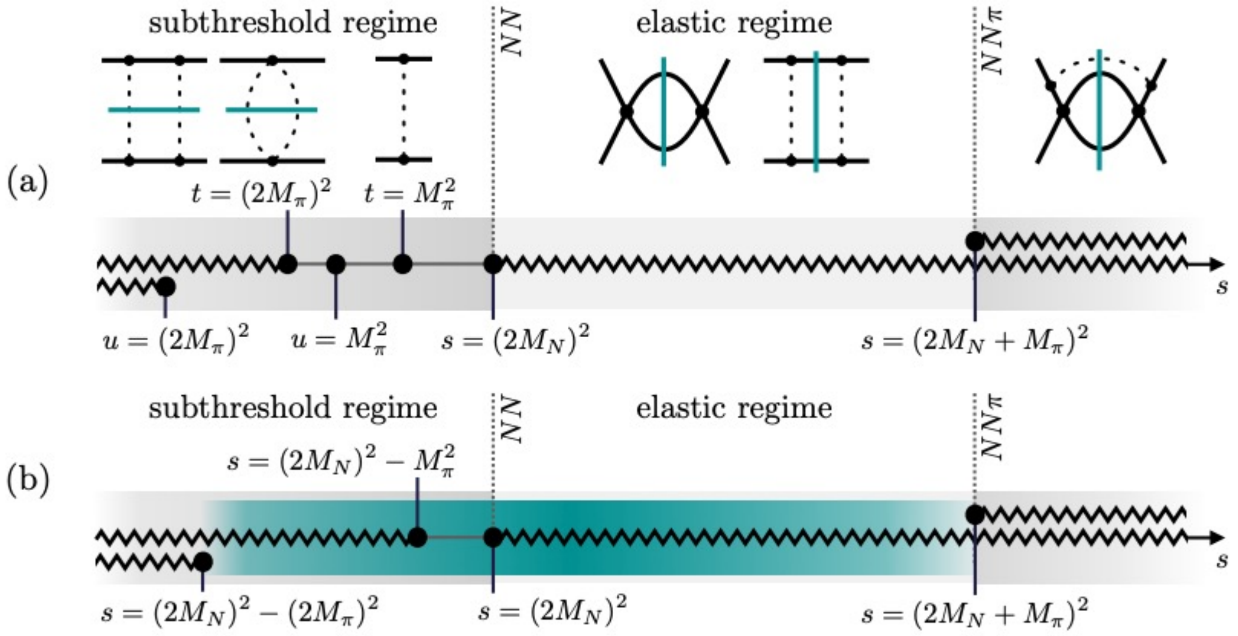
M. Padmanath and S. Prelovsek, 2202.10110

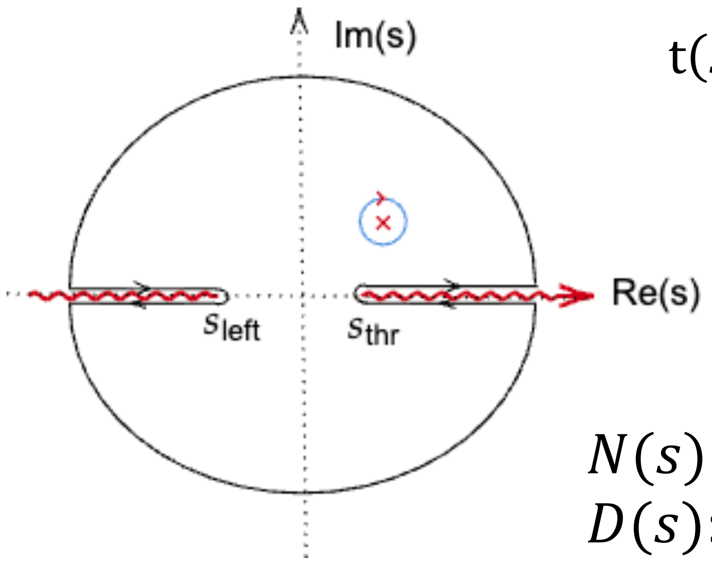


The lhc of OPE limits the application of ERE, which is also used in Lüscher's method to LQCD data

Methods to the LHC problem

A. Baião Raposo and M. T. Hansen, 2303.18793





$$t(s) = \frac{N(s)}{D(s)}$$

$N(s)$: left-hand cuts
$D(s)$: right-hand cuts
Poles: zeros of $D(s)$

➢ Possible solutions:

- EFT + LSE + plane wave basis
  L. Meng et al., JHEP10(2021)051, PRD.109.L071506
- Modified Lüscher + modified integrals/EREs
  A.Baião Raposo et al., JHEP08(2024)075; *JHEP* 06 (2025) 186
  R.Bubna et al., JHEP05(2024)168; JHEP10(2025)197
- Three-body quantization
  M.T.Hansen et al., JHEP06(2024)051; JHEP01(2025)060
- $N\,/D$ representation
  S.M.Dawid, PLB.2025.139442

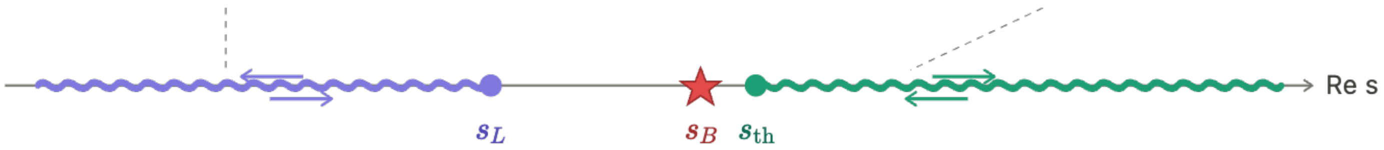
$$k_{\text{os}}^\star \cot\delta_\ell(k_{\text{os}}^\star) = -\frac{1}{a_\ell (k_{\text{os}}^\star)^{2\ell}}\left[1+\sum_{j=1}^{\infty} c_j (k_{\text{os}}^\star)^{2j}\right] + \sum_{n=1}^{P}\sum_{j=0}^{\infty} \frac{A_{n,j}(k_{\text{os}}^\star)^j}{(k_{\text{os}}^\star)^2 - C_n}\,.$$

☞ Our first application to H-dibaryon

N/D → ERE with a LHC

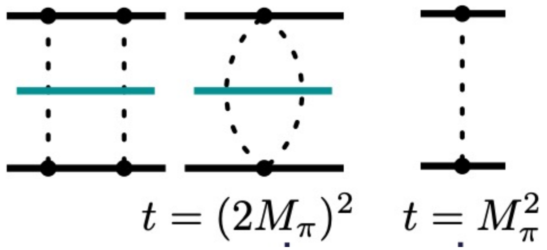
Left-hand cut
$\mathrm{Im}\,N(s) = D(s)\,\mathrm{Im}\,T_L(s), \quad \mathrm{Im}\,D(s) = 0$

Right-hand cut
$\mathrm{Im}\,D(s) = -\rho(s)\,N(s), \quad \mathrm{Im}\,N(s) = 0$



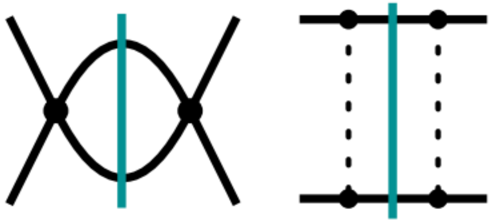
subthreshold regime

Chiral dynamics



→ $\mathrm{Im}\ T_L(s)$

elastic regime



→
- No $\mathrm{Im}\ T_L(s)$
- FV corrections

- ERE:

$$t(s) = \frac{1}{[K(s)]^{-1} - i\rho(s)} = \frac{1}{\rho(s)\cot\delta(s) - i\rho(s)}$$

- ERE with a Chew-Mandelstam term

$$t(s) = \frac{1}{\rho(s)\cot\delta(s) + I(s)}$$

☞ N/D → ERE CM : $N(s) = n_0$

- N/D or ERE with lhc:

$$N(s) = N_m(s) + \frac{s^m}{\pi}\int_{-\infty}^{s_L}\frac{D(s')\mathrm{Im}T_L(s')}{s'^m(s'-s-i\epsilon)}ds'$$

$$D(s) = D_n(s) - \frac{(s-s_0)^n}{\pi}\int_{s_{th}}^{\infty}\frac{\rho(s')N(s')}{(s'-s_0)^n(s'-s-i\epsilon)}ds'$$

OPE: $L_t(s) = -\frac{1}{s-s_{th}}\log\frac{s-s_L}{m_\pi^2}$

→

$$N(s) = n_0 + n_1 s + \cdots + n_m s^m + g_0^2 L_t(s)$$
$$D(s) = 1 + d_1 s + \cdots + d_n s^n$$
$$+\text{once} - \text{subtraction integral}$$

How effective?

Benchmark of N/D: NN case

$$\frac{1}{f_{[m,n]}(p^2)} = \frac{\sum_{i=0}^{n} \tilde{d}_i\, p^{2i} - \tilde{g} d^R(p^2)}{1 + \sum_{j=1}^{m} \tilde{n}_j p^{2j} + \tilde{g}(L(p^2) - L_0)} - ip$$

For different circumstances, the N/D formula can capture the amplitudes,

- Extremely well for virtual states
- Amplitude zeros or poles in $p\cot\delta$ (CDD poles)
  - ☞ below (Halo Nuclei, Efimov) or above threshold (compact states)

W.T.H. van Oers, J.D. Seagrave, PLB 24 (1967) 562
M. T. Yamashita, T. Frederico, L. Tomio, PLB 670 (2008) 49
X.Zhang et al., PRC.108.044304
M.L.Du et al., PRL135 (2025) 1, 011903

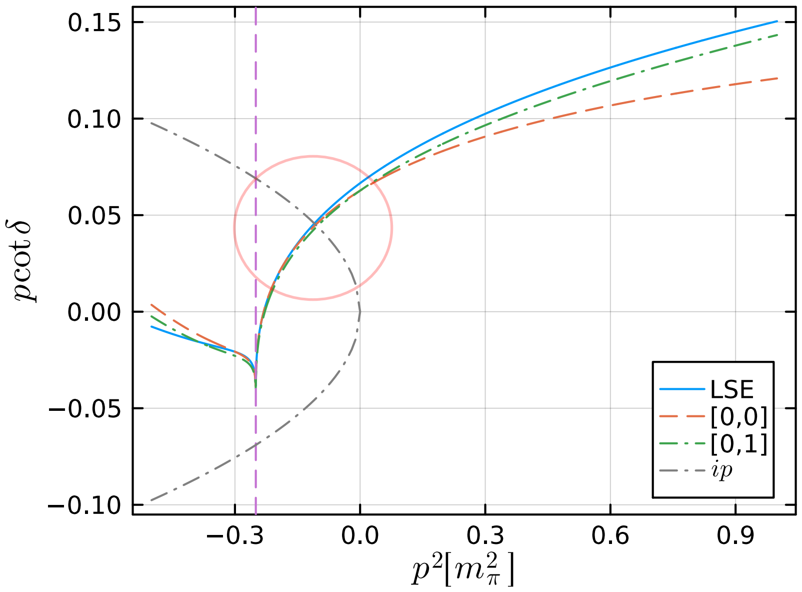
✗ $-\frac{1}{a} + \frac{1}{2} r p^2 + v p^4 + \cdots$

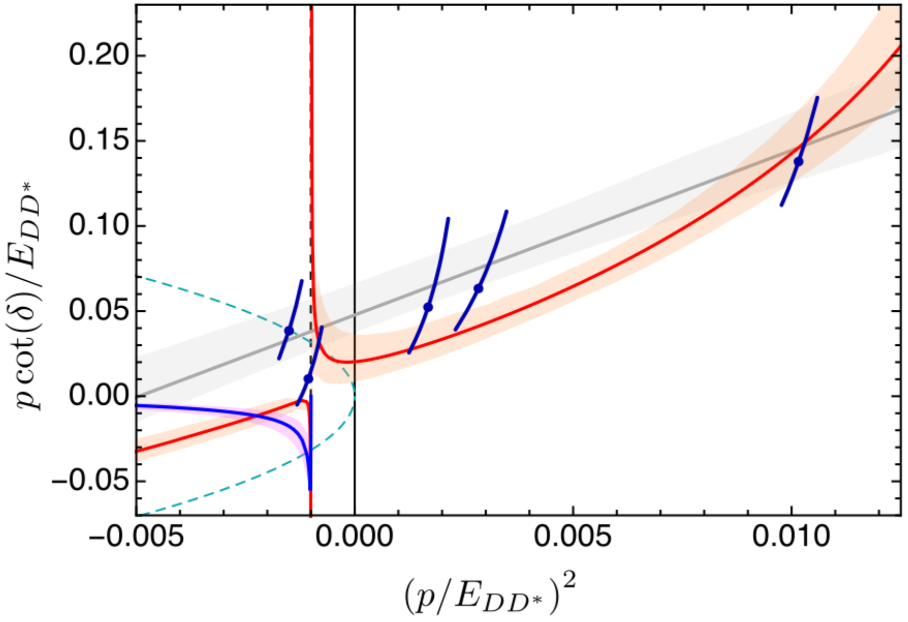
✓ $\frac{d_0 - \tilde{g} d^R(p^2)}{1 + \tilde{g}(L(p^2) - L_0)}$

Short-range + long-range forces

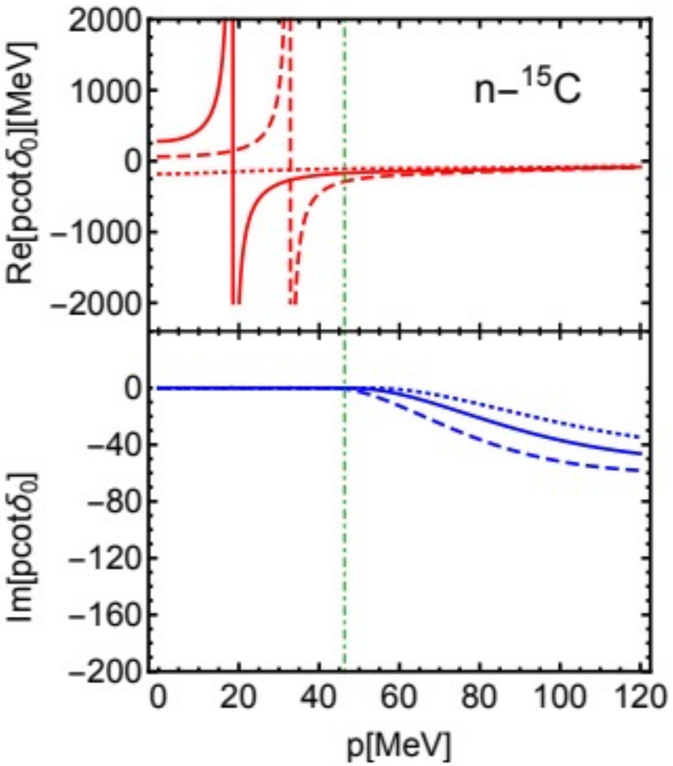
$$1 + \tilde{g}\left(L(p^2) + \frac{1}{m_\pi^2}\right) = 0$$

$$p\cot\delta = -A + Bp^2 - \frac{C}{1 + Dp^2}$$



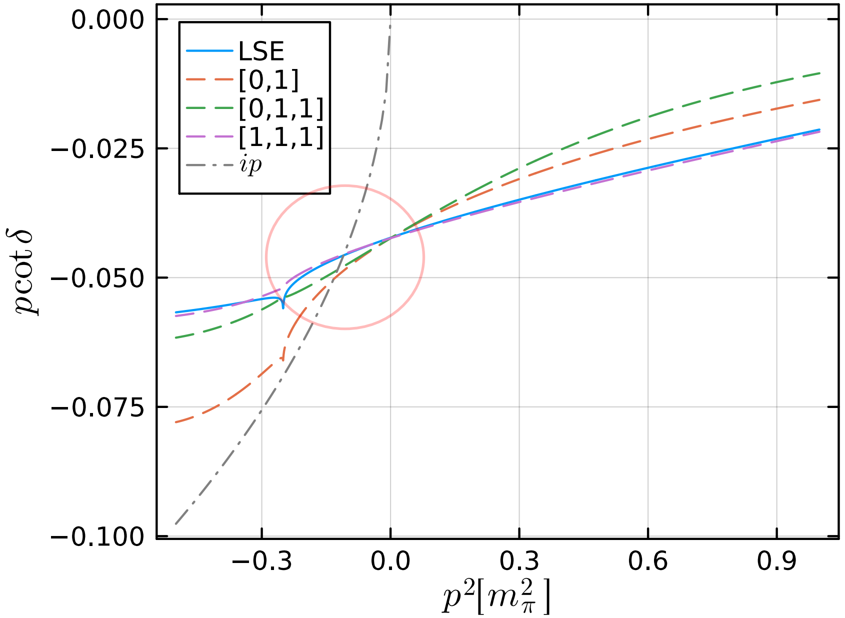


M.-L. Du et al., PRL 131 (2023) 131903



Benchmark of N/D: NN case

M.L.Du et al., PRL135 (2025) 1, 011903
W.J.Wang et al., arXiv:2601.04989



ERE works a bit…

$$\frac{1}{f_{[m,n]}(p^2)} = \frac{\sum_{i=0}^{n} \tilde{d}_i\, p^{2i} - \tilde{g} d^R(p^2)}{1 + \sum_{j=1}^{m} \tilde{n}_j p^{2j} + \tilde{g}(L(p^2) - L_0)} - ip$$

For bound states, the N/D formula

- works well for shallow bound state with tiny OPE coupling
- for deeply bound or large OPE coupling
  - ☞ Discrepancy of imaginary parts of amplitudes
  - ☞ higher-order corrections in N or OPE coupling $\tilde{g} \to g_0 + g_1 p^2 + \cdots$

	Deuteron	**H-dibaryon**
	$[uud][udd], I = 0, J = 1, S = 0$	$uuddss, I = 0, J = 0, S = -2$
At physical masses	$g_{\pi NN}^2/4\pi = 13.5$	Zero $g_{\pi\Lambda\Lambda}$ in SU(3) breaking limit
Binding	Strongly m_π-dependent	Weakly m_π-dependent
On Lattice $\sim e^{-m_\pi L}$	Must study near $m_\pi = 140$MeV $L > 6$fm Signal/noise: $\approx e^{-2(m_N - 3/2 m_\pi)\tau}$	Accessible at heavier m_π Works at $SU(3)_F$:$m_\pi = m_K = m_\eta$ LHC arises from K, η

LQCD history of deuteron

Following André Walk-Loud's talk

"The simplest nucleus is the hardest for LQCD"

A lot of problems

- *hexaquark operator artifact*
- *Signal-to-noise collapse*
- *Unphysical large pion masses*
- *Small volumes*
- *No continuum extrapolation*
- *No FV analysis with LHC*
- …

Get started with H-dibaryon!

➢ LQCD results with (deeply) bound di-nucleons

2006 NPLQCD	—	First dynamical LQCD calc. of NN scattering lengths; no binding claimed
2011 NPLQCD	Mπ ≈ 390 MeV	"Weak evidence" bound deuteron + dineutron; B_d = 11(5)(12) MeV HX ops
2012 Yamazaki et al.	Mπ ≈ 510 MeV	B≤4 nuclei all bound; dineutron also bound — inconsistent with nature wall src
2012 NPLQCD	Mπ ≈ 800 MeV	SU(3) symmetric limit; light nuclei + hypernuclei bound; deep binding claimed — later conclusively refuted HX ops
2015 Yamazaki et al.	Mπ ≈ 310 MeV	Lighter quark mass; deuteron + dineutron still bound wall src
2015 NPLQCD	Mπ ≈ 450 MeV	Phase shifts + ERE extracted; $B_d = 14.4^{+3.2}_{-2.6}$ MeV HX ops
2017 NPLQCD (Wagman et al.)	Mπ ≈ 806 MeV	SU(3) symmetric limit; S-wave BB phase shifts; SU(6) spin-flavor symmetry confirmed; PRD 96, 114510 HX ops
2020 NPLQCD	Mπ ≈ 450 MeV	Axial charge + electroweak matrix elements using same ensembles HX ops
2026 Chakraborty et al.	Mπ = 140–700 MeV	Physical sea quarks; deuteron qualitatively bound, dineutron unbound at physical point new

➢ LQCD results without bound di-nucleons (or inconclusive)

2012 HAL QCD (Ishii et al.)	Mπ ≈ 700 MeV	Time-dependent NBS potential method introduced; no NN bound state; nuclear force attractive but insufficient; PLB 712, 437 potential
2012 HAL QCD (Inoue et al.)	Mπ = 469–1171 MeV	Coupled-channel BB interactions; forces become weaker at heavier Mπ; Nucl. Phys. A881, 28
2016 HAL QCD (Iritani et al.)	Mπ ≈ 510 MeV	"Mirage plateau": HX operators produce fake plateaus mimicking bound states; JHEP 10 (2016) 101 key critique
2019 "Mainz"	Mπ ≈ 960 MeV	Staggered fermions; no NN binding at very heavy quark mass
2020 CoSMoN	Mπ ≈ 714 MeV	Stochastic LapH diffuse ops; no bound dineutron; first direct-method support for HAL QCD diffuse ops
2021–23 NPLQCD	Mπ ≈ 800 MeV	Wide variational operator basis; operator-dependent spectra; self-critical retraction inconclusive
2025 BaSc	Mπ ≈ 714 MeV	Momentum-space + HAL QCD on same ensembles: no bound di-nucleon; controversy resolved new

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Introduction to H-dibaryon

- H-dibaryon, a hypothetical six-quark ($uuddss$) bound state
  - ☞ Flavor-singlet, $J^P = 0^+, I = 0, S = -2$
  - ☞ $M_H \approx 2150\text{MeV}$, $E_B \approx 80\text{MeV}$ Predicted by Jaffe within MIT bag model

Jaffe 1977, PRL 38.195

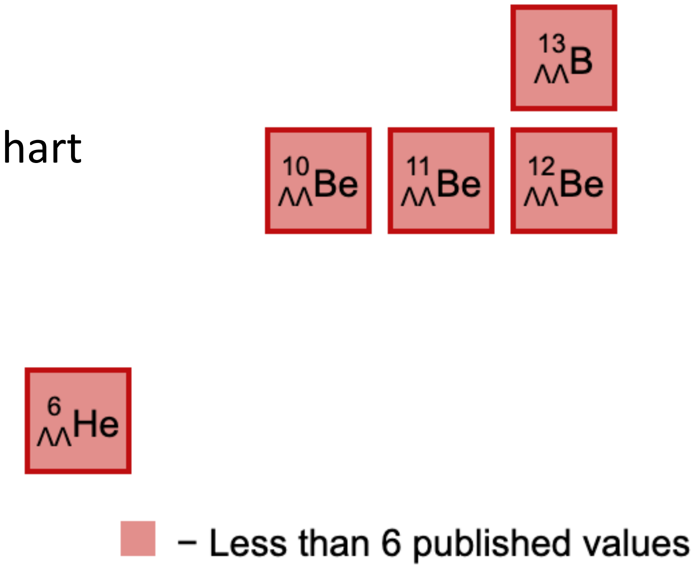
- The $\Lambda\Lambda, N\Xi, \Sigma\Sigma$ $(\mathrm{I}=0)$ channels form the singlet via:

Inoue et al. (HAL QCD) 2010, 10.1143/PTP.124.591]

$$\begin{pmatrix} \langle\Lambda\Lambda| \\ \langle\Sigma\Sigma| \\ \langle N\Xi| \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{27}{40}} & -\sqrt{\frac{8}{40}} & -\sqrt{\frac{5}{40}} \\ -\sqrt{\frac{1}{40}} & -\sqrt{\frac{24}{40}} & \sqrt{\frac{15}{40}} \\ \sqrt{\frac{12}{40}} & \sqrt{\frac{8}{40}} & \sqrt{\frac{20}{40}} \end{pmatrix} \begin{pmatrix} \langle\mathbf{27}| \\ \langle\mathbf{8}_s| \\ \langle\mathbf{1}| \end{pmatrix}$$

H.Ekawa et al., arXiv:1811.07726
https://hypernuclei.kph.uni-mainz.de/

- H-dibaryon helps to
  - study the hyperon-hyperon (nucleon) interactions ☞ double-Λ hypernuclear chart
  - study strange matter, neutron star EOS, dark matter candidate



Bombaci 2017, JPS Conf. Proc. 17, 101002
Leong, Thomas & Guichon 2025, arXiv:2503.21171v1
Farrar 2017, arXiv:1708.08951v2
Kolb & Turner 2019, PRD 99.063519
Azizi, JPhysG 47, 095001 (2020)

Experimental search of H-dibaryon

1985–1994 BNL-AGS E813/E836	BNL, K^- beam	$K^-+{}^3He \to K^+ + H + n$; upper limits 50–380 MeV/c^2 below ΛΛ; ~10× below theory no signal
1996 BNL-AGS E888	BNL, neutral beam	$H \to \Lambda n$, $H \to \Sigma^0 n$ weak decay; $(d\sigma_H/d\Omega)/(d\sigma_\Lambda/d\Omega) < 6.3\times10^{-6}$ at 90% CL no signal
1997–2001 BNL-AGS E906	BNL, Ξ^- capture	ΛΛ correlations; initial H claim later downgraded; $M_H > 2223$ MeV/c^2 at 90% CL downgraded
2001 KEK-PS E373 (NAGARA)	KEK, nuclear emulsion	Unique ID of ${}^6_{\Lambda\Lambda}He$; $B_{\Lambda\Lambda}$=6.91±0.16 MeV → $M_H > 2m_\Lambda - 7.13$ MeV = 2223 MeV/c^2 key constraint
2007 KEK E522 (Yoon et al.)	KEK, K^- beam	${}^{12}C(K^-,K^+\Lambda\Lambda X)$ at 1.8 GeV/c; bump near ΛΛ threshold seen but no excess above cascade model+FSI; $d\sigma_H/d\Omega_L < 2.1\pm0.6_{stat}\pm0.1_{syst}$ μb/sr at 90% CL; PRC 75, 022201(R) no signal
2011 STAR (RHIC)	RHIC, Au+Au	ΛΛ correlations at $\sqrt{s_{NN}}$=39, 62, 200 GeV; $H \to \Lambda p \pi^-$ weak decay search; no signal no signal
2013 Belle	KEKB, e^+e^-	Υ(1S,2S) decays; $H \to \Lambda p \pi^-$ and ΛΛ; 90% CL UL 1–2 orders below B(Υ→d); 102M+158M events no signal
2015 ALICE (LHC)	LHC, Pb-Pb 2.76 TeV	$H \to \Lambda p \pi^-$ in central (0–10%) Pb-Pb; 99% CL UL across full lifetime range vs thermal model no signal
2019 J-PARC E07	J-PARC, nuclear emulsion	${}_{\Lambda\Lambda}Be$ event (Mino) via Ξ^- capture in ${}^{16}O$; most likely $B_{\Lambda\Lambda}$=19.07±0.11 MeV (${}^{10}_{\Lambda\Lambda}Be$ and ${}^{12}_{\Lambda\Lambda}Be^*$ also possible); ΛΛ interaction constrained new event
2021–2026 J-PARC E42	J-PARC, HypTPC	${}^{12}C(K^-,K^+)$ at p_{K^-}=1.8 GeV/c; 0.3M events (2021); ΛΛ, $\Lambda p\pi^-$, $\Xi^- p$ spectra; prelim. MENU 2023; final analysis ongoing preliminary
2025 J-PARC E07 (AI)	J-PARC, deep learning	First new double-Λ event (${}_{\Lambda\Lambda}13$ B) in 25 years; 0.2% of data → >2000 S=−2 events estimated in full dataset new

No evidence for deeply-bound H

H.Takahashi et al., PRL87.212502

$\checkmark\ {}^6_{\Lambda\Lambda}He \to {}^5_\Lambda He + p + \pi^-$

$\times\ {}^6_{\Lambda\Lambda}He \to {}^4 He + H$

ΛΛ threshold enhancement

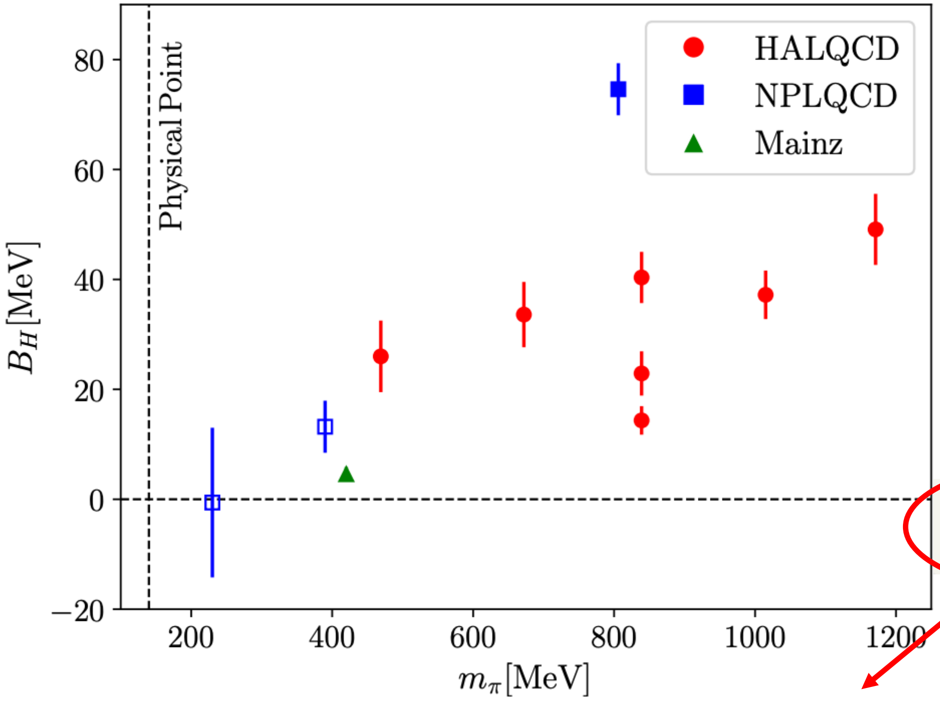
- Deeply-bound:
  MIT bag model, chiral model
- Weakly-bound:
  $H \to \Lambda p \pi^-, \Sigma^- p$; direct
  ΛΛ femtoscopic correlation interactions ~3.2MeV
- Virtual state/resonance:
  ΛΛ or $\Sigma^- p$ threshold effect

LQCD study on H-dibaryon

Studies: heavier-than-physical pion masses, mostly at the $\mathrm{SU}(3)_F$-symmetric point.

- No agreement on $\mathrm{E_B}$ for the same $\mathrm{m_\pi}$
- Discretization errors/spacing dependence

BaSc, arXiv: [2603.00698](2603.00698)



We are here today!

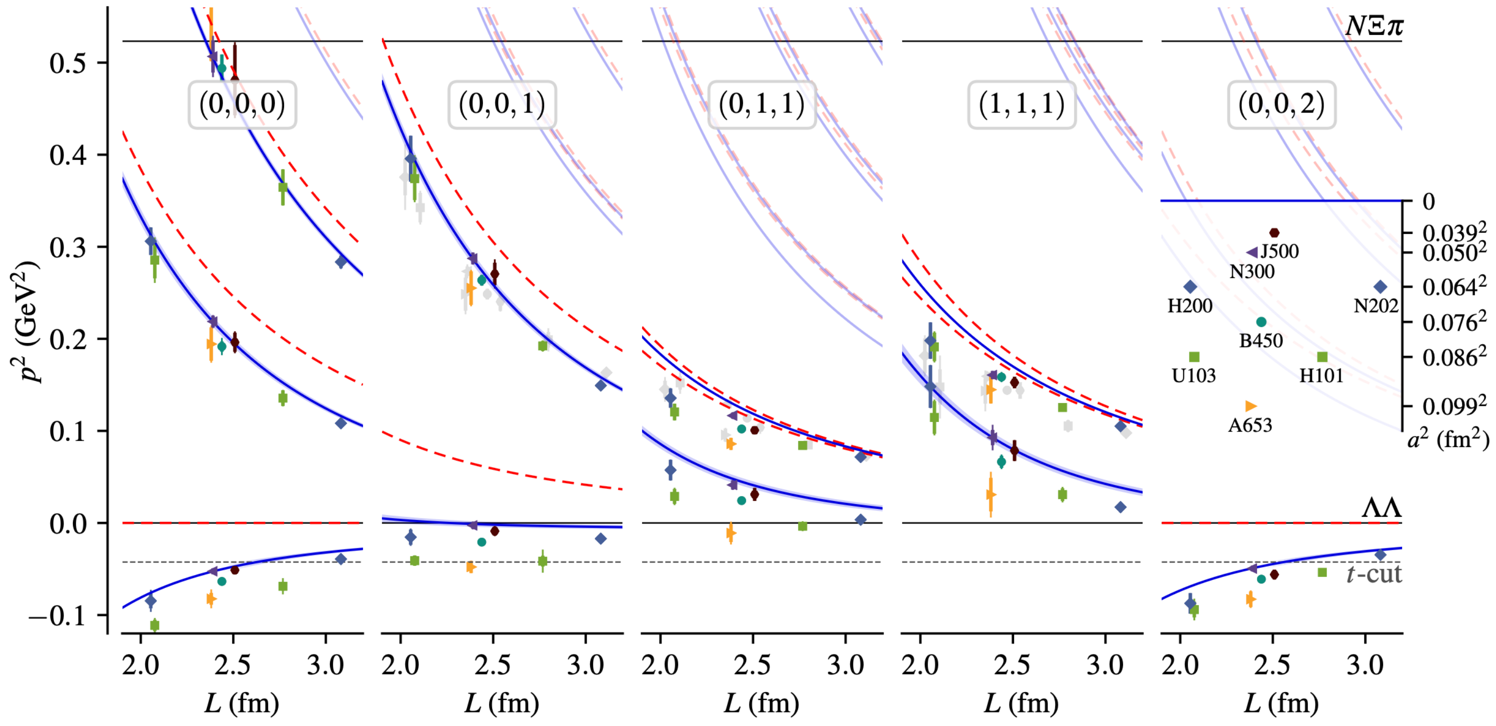
1985 Mackenzie & Thacker	quenched	First LQCD attempt; no bound H-dibaryon found; PRL 55, 2539	no binding
1999 Negele et al. (MIT)	quenched, $16^3\times32$	$M_H \approx 100$ MeV above ΛΛ threshold; finite-volume artifacts overestimate binding on smaller lattice	unbound
2011 NPLQCD (Beane et al.)	$N_f=2+1$, $m_\pi=389$ MeV	First dynamical evidence; $B_H=16.6\pm2.1\pm4.6$ MeV; anisotropic clover, 4 volumes (L=2.0–3.9 fm); PRL 106, 162001	bound
2011 HAL QCD (Inoue et al.)	$N_f=3$, $m_\pi=469$–1171 MeV	NBS potential method; $B_H=20$–50 MeV across SU(3) symmetric point; PRL 106, 162002	bound
2012 HAL QCD (Inoue et al.)	$N_f=3$, SU(3) limit	Coupled-channel ΛΛ–NΞ–ΣΣ; resonance between ΛΛ and NΞ thresholds at physical m_s; Nucl. Phys. A881	resonance
2013 NPLQCD (Beane et al.)	$N_f=3$, $m_\pi=806$ MeV	SU(3) symmetric limit; H + many dibaryons bound; spin-flavor SU(6) symmetry analysis; PRD 87, 034506	bound
2013 Shanahan, Thomas, Young	chiral extrapolation	EFT chiral extrapolation of NPLQCD+HAL results; H unbound by 26±11 MeV above ΛΛ threshold at physical point	likely unbound
2019 Francis, Green et al.	$N_f=2$, quenched s	Hexaquark + bilocal operators; operator dependence studied; H weakly bound; dineutron unlikely bound; PRD 99, 074505	operator study
2021 Green, Hanlon, Junnarkar, Wittig	$N_f=2$, 6 spacings	First continuum limit (a→0); large lattice artifacts found; $B_H=4.56\pm1.13^{+0.63}_{-0.63}$syst MeV at SU(3) pt $m_\pi=m_K\approx420$ MeV; PRL 127, 242003	continuum limit

BaSc, $\mathrm{m_\pi} \approx 280$ MeV, preliminary study of the H dibaryon in $\mathrm{N_f} = 2 + 1$ lattice QCD

☞ ΛΛ, NΞ, ΣΣ correlations

H-dibaryon at SU(3)-flavor-symmetric point

Green et al., PRL127.242003



p^2: C.M. momentum by $\mathrm{E_{cm}} = 2\sqrt{p^2 + m^2}$

Points: lattice energy levels

--------: noninteracting levels

———: interacting levels in the continuum

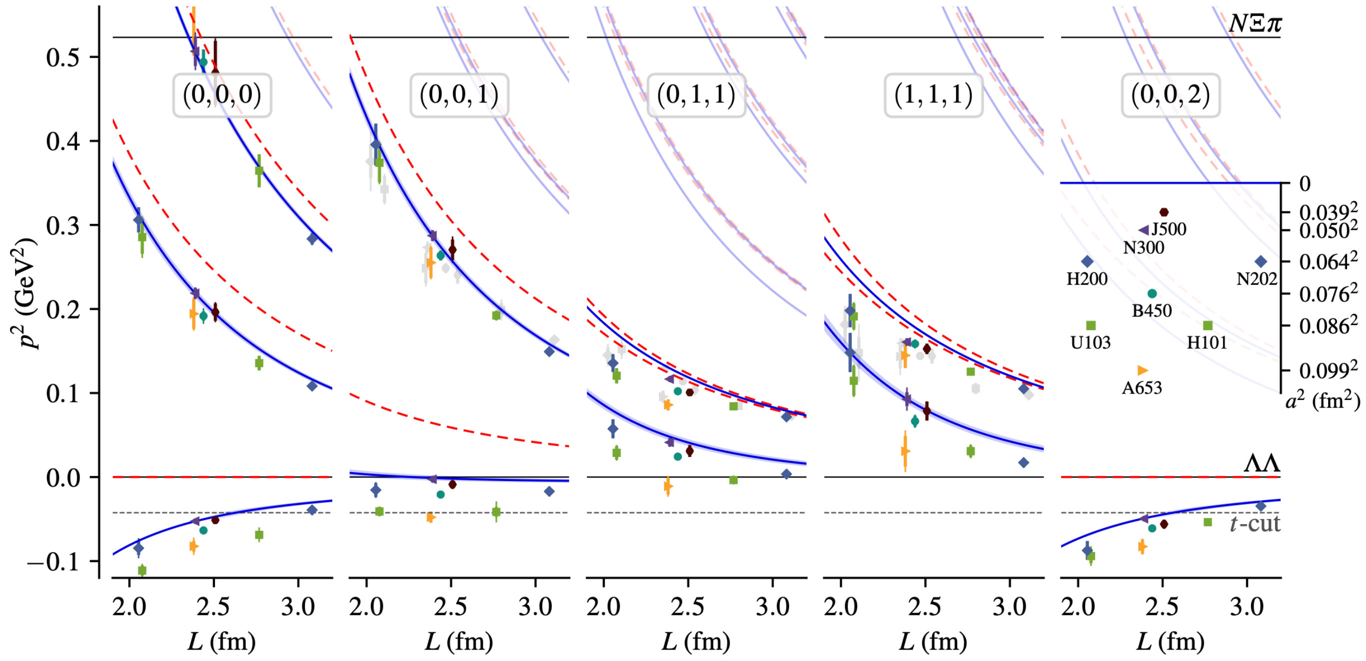
Label	m_π (MeV)	m_B (GeV)
J500	411	1.18
N300	422	1.20
N202	412	1.17
H200	419	1.20
B450	417	1.18
H101	417	1.16
U103	414	1.18
A653	424	1.17

$m_B \approx 1.18\mathrm{GeV}$

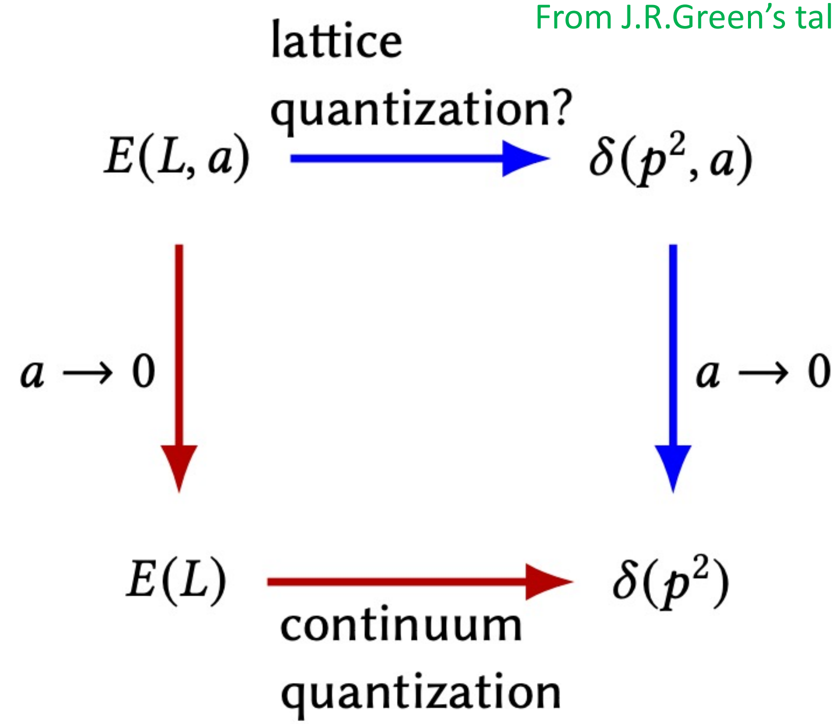
- Ensembles with $\mathcal{O}(\mathrm{a})$ improved Wilson-clover Fermions from CLS
- $\mathrm{SU(3)_F}$-symmetric point with physical $m_u + m_d + m_s$ ☞ $m_\pi = m_K = m_\eta \approx 420\mathrm{MeV}$
- 8 ensembles (L, a) + 5 frames ☞ significant discretization effects
- t-channel left-hand cut: $p^2 = -\frac{m_\pi^2}{4}$

Quantization condition and continuum limit

Green et al., PRL127.242003



From J.R.Green's talk

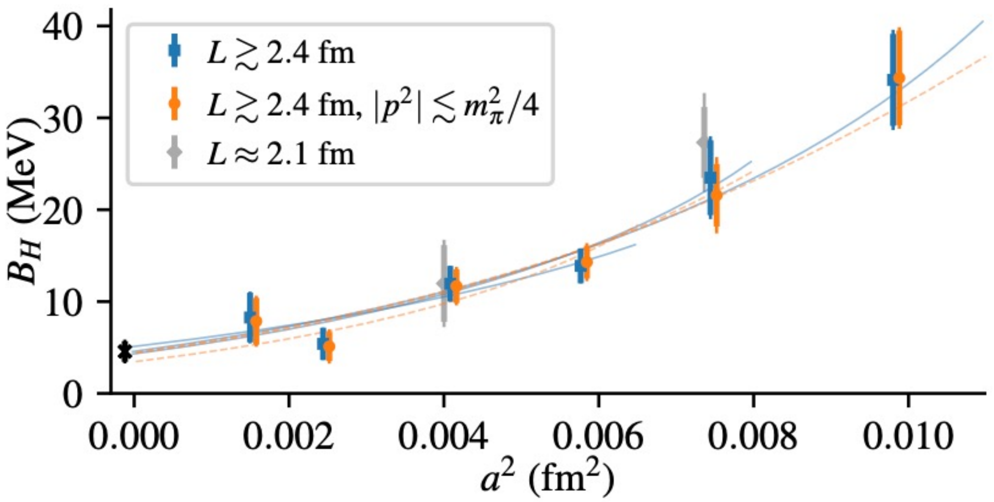


$$p\cot\delta(p) = \sum_{i=0}^{N-1} c_i p^{2i}, \qquad c_i = c_{i0} + c_{i1} a^2$$

- Near-threshold fitting ($|p^2| \le \frac{m_\pi^2}{4}$): N = 2 (ERE)

  ☞ $B_H^{SU(3)_f} = 4.56 \pm 1.13 \pm 0.63$MeV

- Full-range fitting ($p^2 \ge -\frac{m_\pi^2}{4}$): N = 3



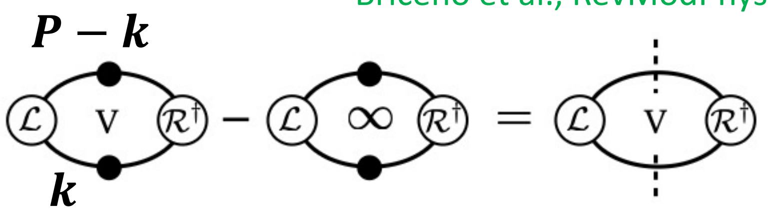
Lüscher's quantization condition

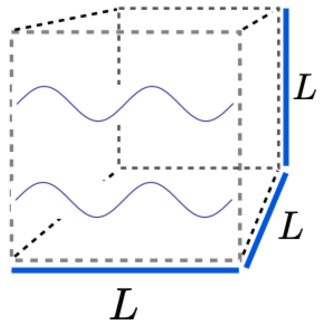
Lüscher, NPB354(1991); C.h.Kim et al., NPB727(2005)
Briceno et al., RevModPhys.90.025001; PRD94013008

➢ Sum-integral difference:

$$\mathcal{F}_L = -\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int \frac{d\mathbf{k}}{(2\pi)^3}\right]\frac{1}{2\omega_k\, 2\omega_{Pk}}$$
$$\times\, \mathcal{L}(P-k,k)\frac{1}{E-\omega_k-\omega_{Pk}+i\epsilon}\mathcal{R}^\dagger(P-k,k)\Big|_{k_4=i\omega_k}$$

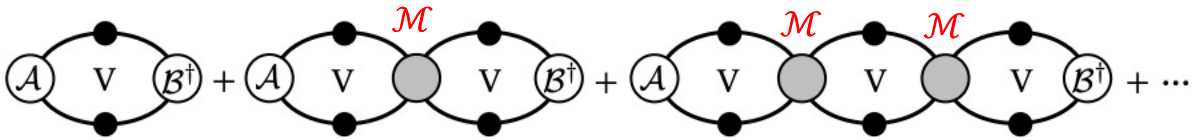
$$\mathcal{F}_L = -\mathcal{L}_{\ell m}(P)\, F_{\ell m;\ell' m'}(P,L)\, \mathcal{R}^\dagger_{\ell' m'}(P),$$





- off-shell part $\sim e^{-m_\pi L}$
- on-shell part survives
- $\mathcal{L}, \mathcal{R}$ smooth function ☞ FV correction $\sim L^{-n}$

➢ Correlation function



two-body cuts → a series of poles ☞ Energy levels

$$C_L(P) = C_\infty(P) - A(P)\frac{1}{F^{-1}(P,L)+\mathcal{M}(P)}B^*(P)$$

➢ Quantization condition

$$\det[F^{-1}(P,L)+\mathcal{M}(P)] = 0\,.$$

$$\det_{\ell m_\ell}\left[\mathcal{K}^{-1}(s) + F(P,L)\right] = 0$$

$$F_{\ell' m'_\ell;\ell m_\ell}(P,L) = \xi\left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \text{p.v.}\int\frac{d\boldsymbol{k}}{(2\pi)^3}\right]\left(\frac{k^\star}{p^\star}\right)^{\ell'}\frac{(4\pi)\,Y^*_{\ell' m_{\ell'}}(\hat{\boldsymbol{k}}^\star)\,Y_{\ell m_\ell}(\hat{\boldsymbol{k}}^\star)}{2\omega_1(\boldsymbol{k})\,2\omega_2(\boldsymbol{P}-\boldsymbol{k})\,(E-\omega_1(\boldsymbol{k})-\omega_2(\boldsymbol{P}-\boldsymbol{k}))}\left(\frac{k^\star}{p^\star}\right)^{\ell},$$

N/D quantization condition

S.M.Dawid et al., PLB864(2025)139442

- $\mathcal{N} = \mathcal{N}_L + \mathcal{O}(e^{-m_\pi L})$

$$\tilde{\mathcal{K}}^{-1}(p^\star) = \underbrace{\left[\mathbb{1} - \xi\,\text{p.v.}\int \frac{\mathrm{d}\boldsymbol{k}^\star}{(2\pi)^3}\,\mathcal{L}(\boldsymbol{k},\boldsymbol{P})\,\frac{\mathcal{Y}^*(\boldsymbol{k}^\star)\otimes\mathcal{Y}(\boldsymbol{k}^\star)^T\mathcal{N}(k^\star)}{E^\star(\boldsymbol{k})^2 - s}\right]}_{D_{\text{IFV}}}\mathcal{N}^{-1}(p^\star)$$

$$\det_{\ell m_\ell}\left[\tilde{\mathcal{K}}^{-1}(s) + \tilde{F}(P,L)\right] = 0$$

$$\tilde{F}_{\ell' m'_\ell;\ell m_\ell}(P,L) = \xi\left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \text{p.v.}\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\right]\frac{(4\pi)\,(k^\star)^{\ell'}\,Y^*_{\ell' m_{\ell'}}(\hat{\boldsymbol{k}}^\star)\,Y_{\ell m_\ell}(\hat{\boldsymbol{k}}^\star)\,(k^\star)^{\ell}}{2\omega_1(\boldsymbol{k})\,2\omega_2(\boldsymbol{P}-\boldsymbol{k})\,\left(E - \omega_1(\boldsymbol{k}) - \omega_2(\boldsymbol{P}-\boldsymbol{k})\right)}$$

- $\tilde{F}$ only picks the pole and residue

$$\not{\Sigma}\,\frac{1}{E - \omega_1(k) - \omega_2(P-k)} = \not{\Sigma}\,\frac{2E}{E^2 - \left(\omega_1(k) + \omega_2(P-k)\right)^2} + \not{\Sigma}\,\text{smooth function}$$

➢ Quantization condition: $D_{\text{FV}} = 0$

$$\det\left[\mathbb{1} + \frac{\xi}{L^3}\sum_{\boldsymbol{k}}\mathcal{L}(\boldsymbol{k},\boldsymbol{P})\,\frac{\mathcal{Y}^*(\boldsymbol{k}^\star)\otimes\mathcal{Y}(\boldsymbol{k}^\star)^T}{s - E^\star(\boldsymbol{k})^2}\,\mathcal{N}(k^\star)\right] = 0 \qquad \mathcal{L}(k,P) = \frac{2E(k)}{2\omega_1(k)2\omega_2(P-k)}$$

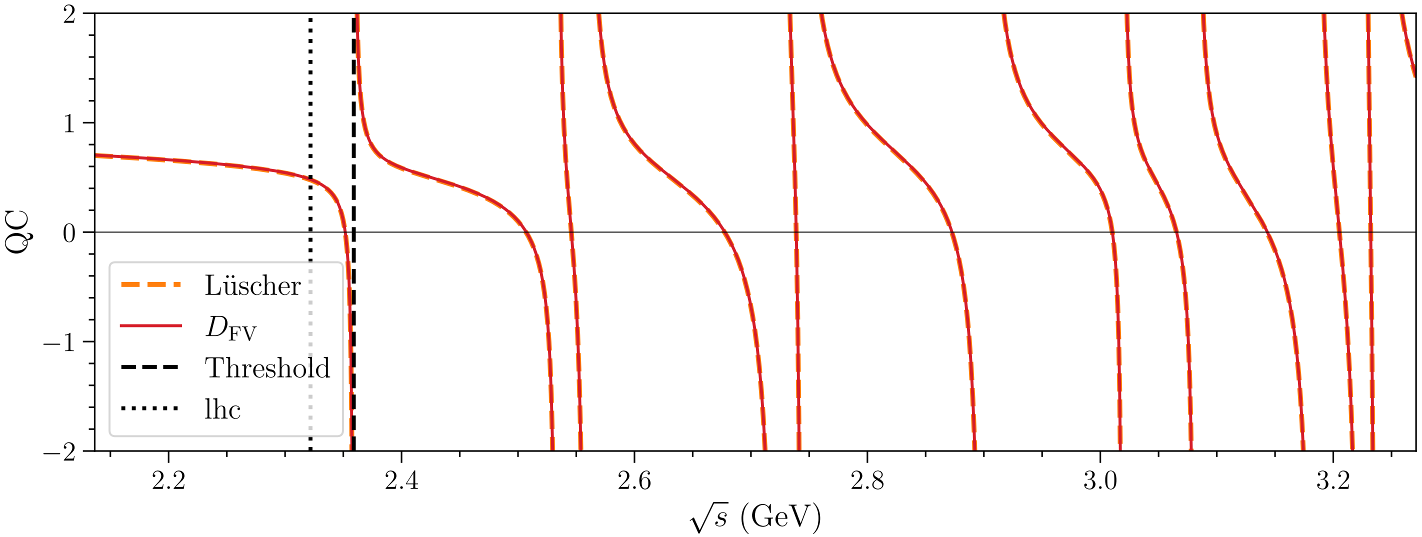
$+p_n(s) + \text{CDD poles}$

Comparison between QCs

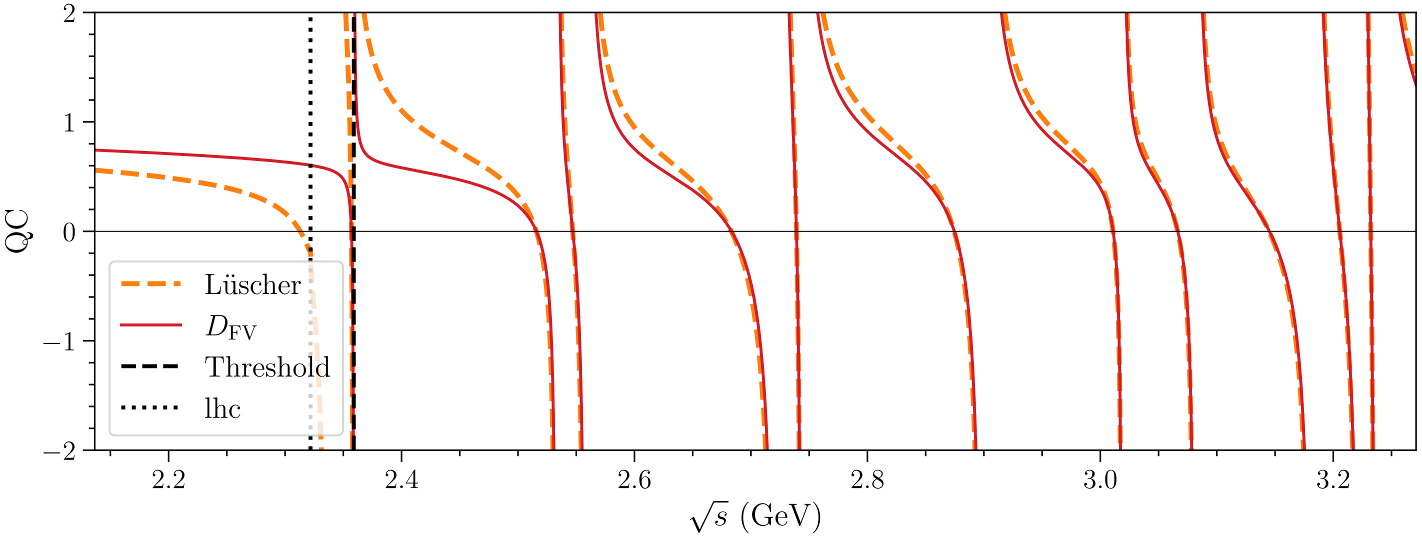
The two quantization conditions are equivalent above lhc.

$$\frac{N(s)}{D(s)} \quad v.s. \quad \frac{1}{[K(s)]^{-1} - i\rho(s)}$$

$N(s) = 1$

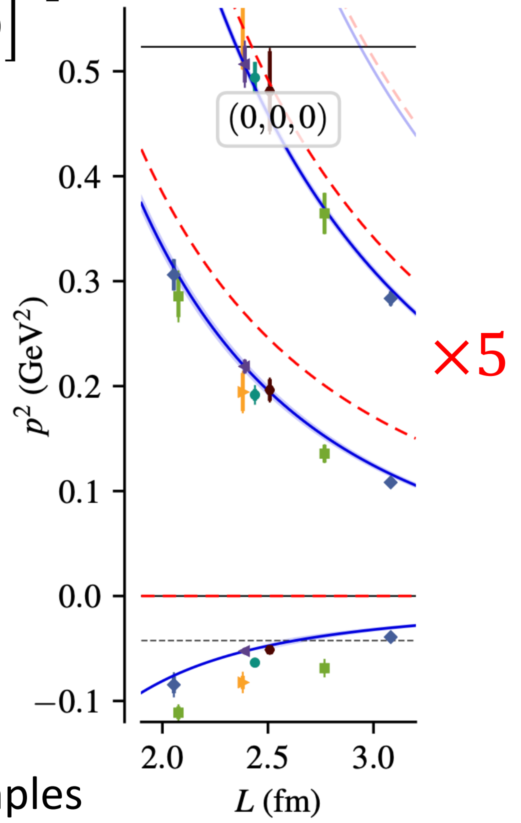


$N(s) = 1 + 0.3^2 L_t(s)$



Fitting strategies

- **Lüscher's QC**: $K^{-1}(E) + F(E,P,L) = 0$
  - ERE: $t(s) = \left[\frac{2}{\sqrt{s}}\left(c_0 + c_1[p(s)] + c_2[p(s)]^2\right) - i\rho(s)\right]^{-1}$
  - ERE CM: $t(s) = \left[\frac{2}{\sqrt{s}}\left(c_0 + c_1[p(s)] + c_2[p(s)]^2\right) + I(s)\right]^{-1}$
- **N/D QC**:
  - $N(s) = n_0 + g_0^2 L_t(s)$
  - $D_{FV}(s,P) = 1 + d_0 s + d_1 s^2 - \frac{16\pi}{L^3}\sum_k L(k,P)\frac{s}{E^*(k)^2}\frac{N(k^*,P)}{E^*(k)^2 - s}$
- **Fitting range**: lowest energy level → three-body threshold (covering lhc)
- **spacing effect?**
  - w/o spacing dependence: parameter ξ_i
  - w/ spacing dependence: parameter $\xi_i = \xi_{i0} + \xi_{ia}a^2$
- **61 data points**: $\chi^2 = \sum_{i,j}\left(p_i^2 - p_{\mathrm{FV},i}^2\right)\Sigma_{ij}^{-1}\left(p_j^2 - p_{\mathrm{FV},j}^2\right)$
  - $\Sigma_{\mathrm{stat},ij}$: from 10^3 boostrapping samples
  - $\Sigma_{\mathrm{syst},ij} = (\delta p^2)_i(\delta p^2)_j$



Predicted spectrum w/o a-dependence

orange dashed: ERE

green dotted: ERE CM

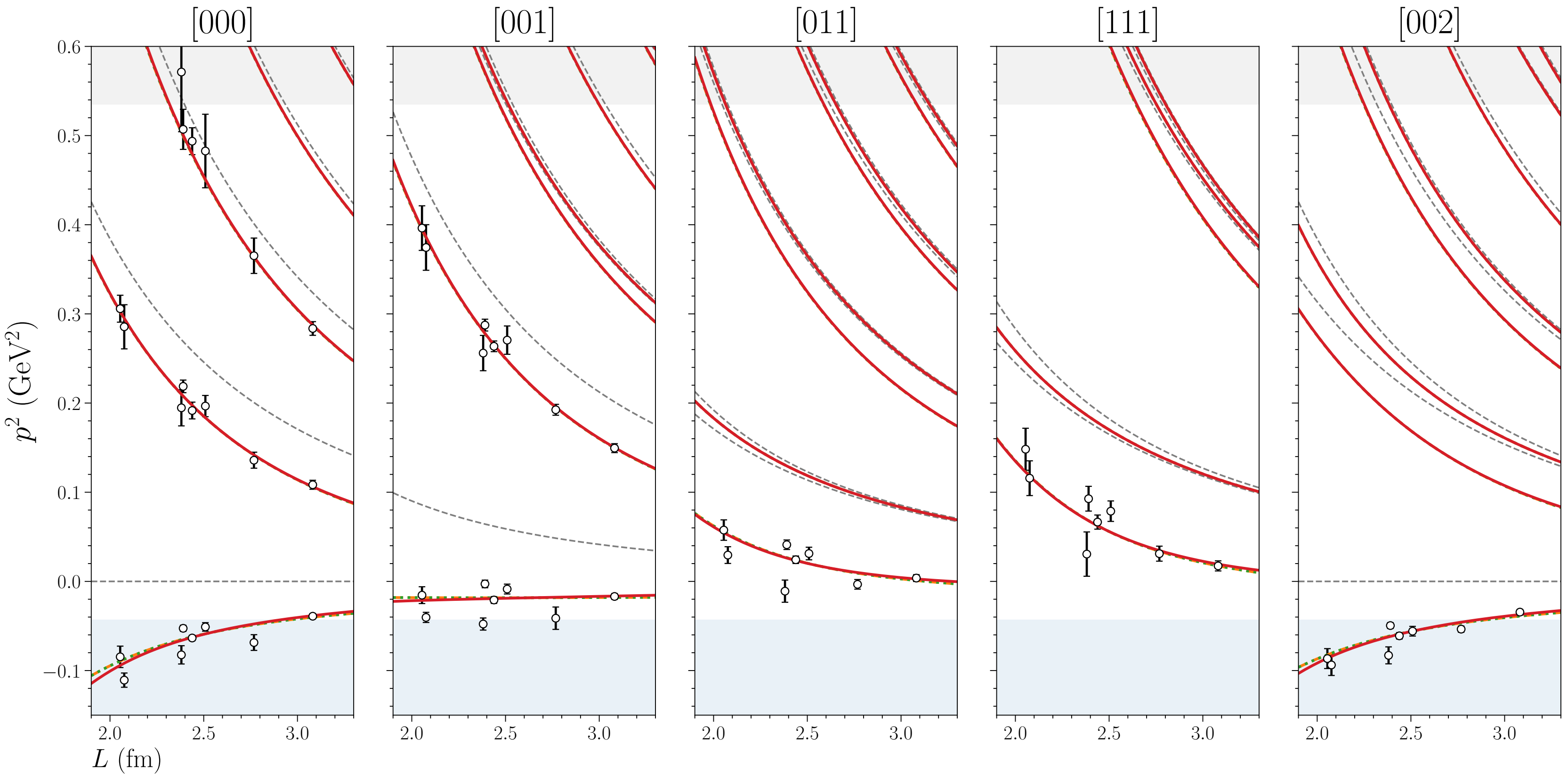
red solid: N/D

(c_0, c_1, c_2)

(n_0, g_0, d_1, d_2)

$1.62 < 1.65$

- better χ^2/ndof in N/D
  - ☞ only differ in the lowest levels
  - ☞ mild left-hand-cut effect
- significant L dependence
- largely bound ☞ spacing effect



Predicted spectrum w/ a-dependence

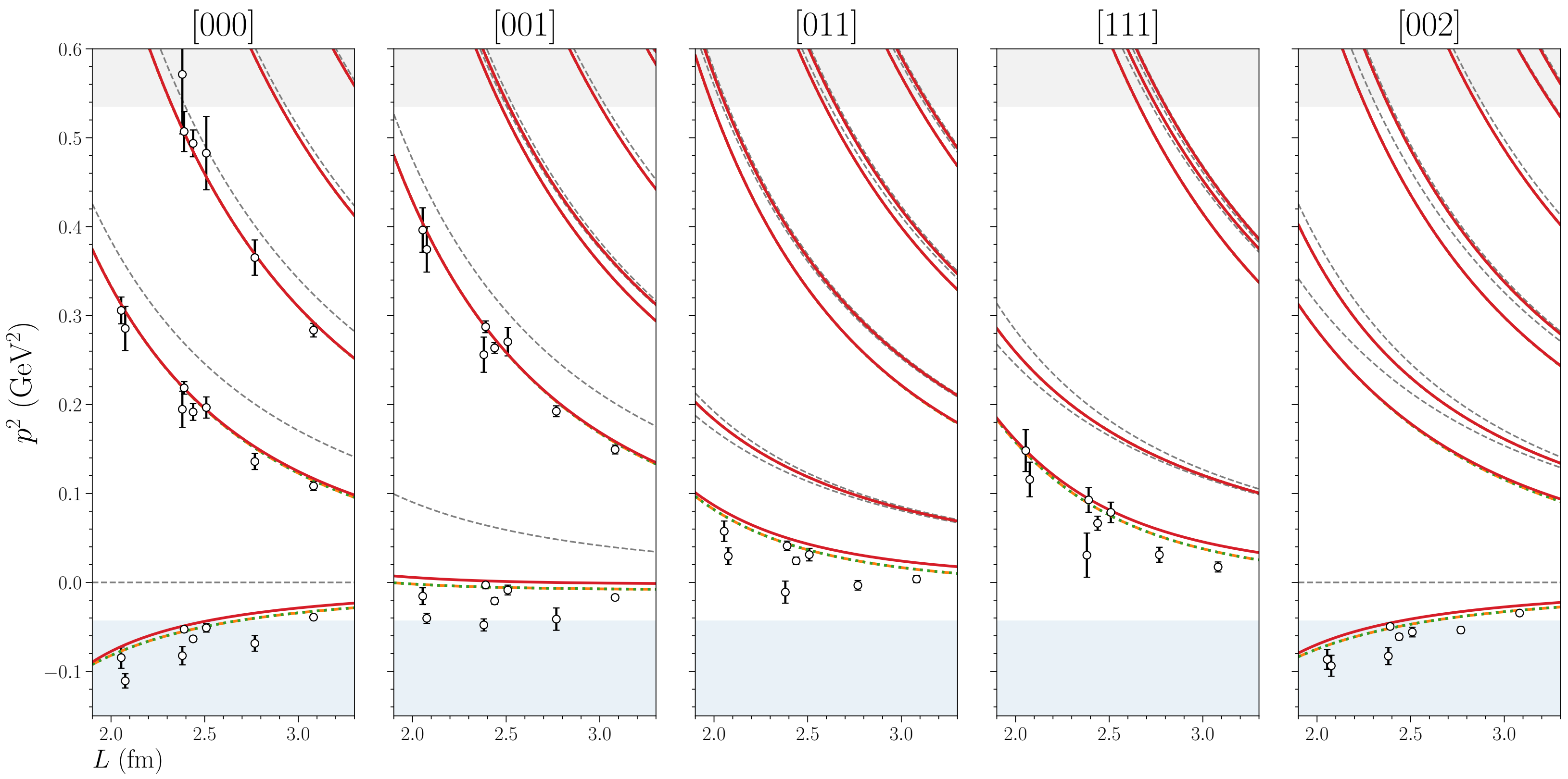
orange dashed: ERE $\sharp\ (c_{00}, c_{0a}, c_{10}, c_{1a}, c_{20}, c_{2a})$

green dotted: ERE CM

red solid: N/D $\sharp\ (n_{00}, n_{0a}, g_{00}, g_{0a}, d_{10}, d_{1a}, d_{20}, d_{2a})$

$0.7615 < 0.7622$

- Slightly better χ^2/ndof in N/D fitting
- Spectrum in the continuum ($a = 0$) ☞
- comparable parameters at $\mathcal{O}(a^0)$
  - ☞ comparable observables
  - ☞ shallow bound state



Cross-check on continuum extrapolation

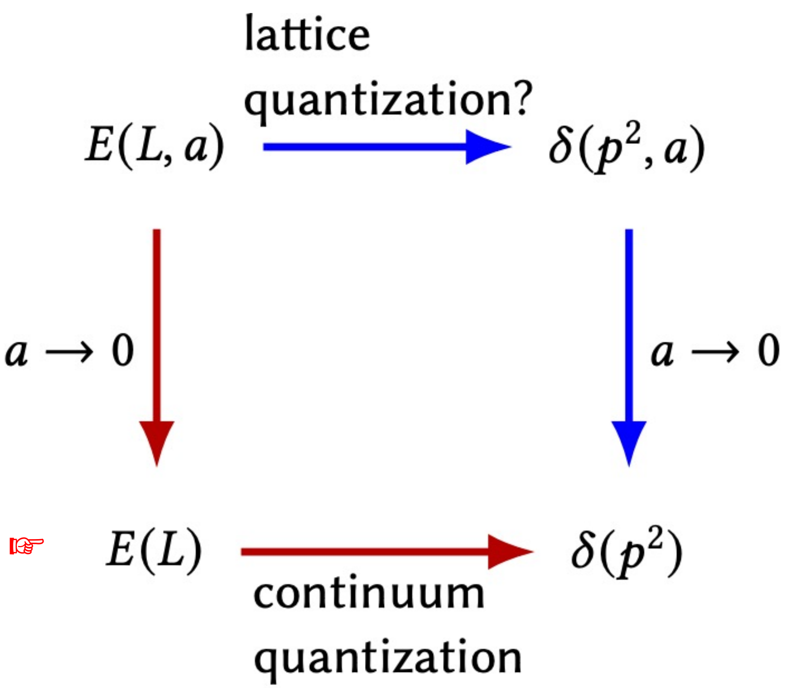
orange dashed: ERE # (c_0, c_1, c_2)

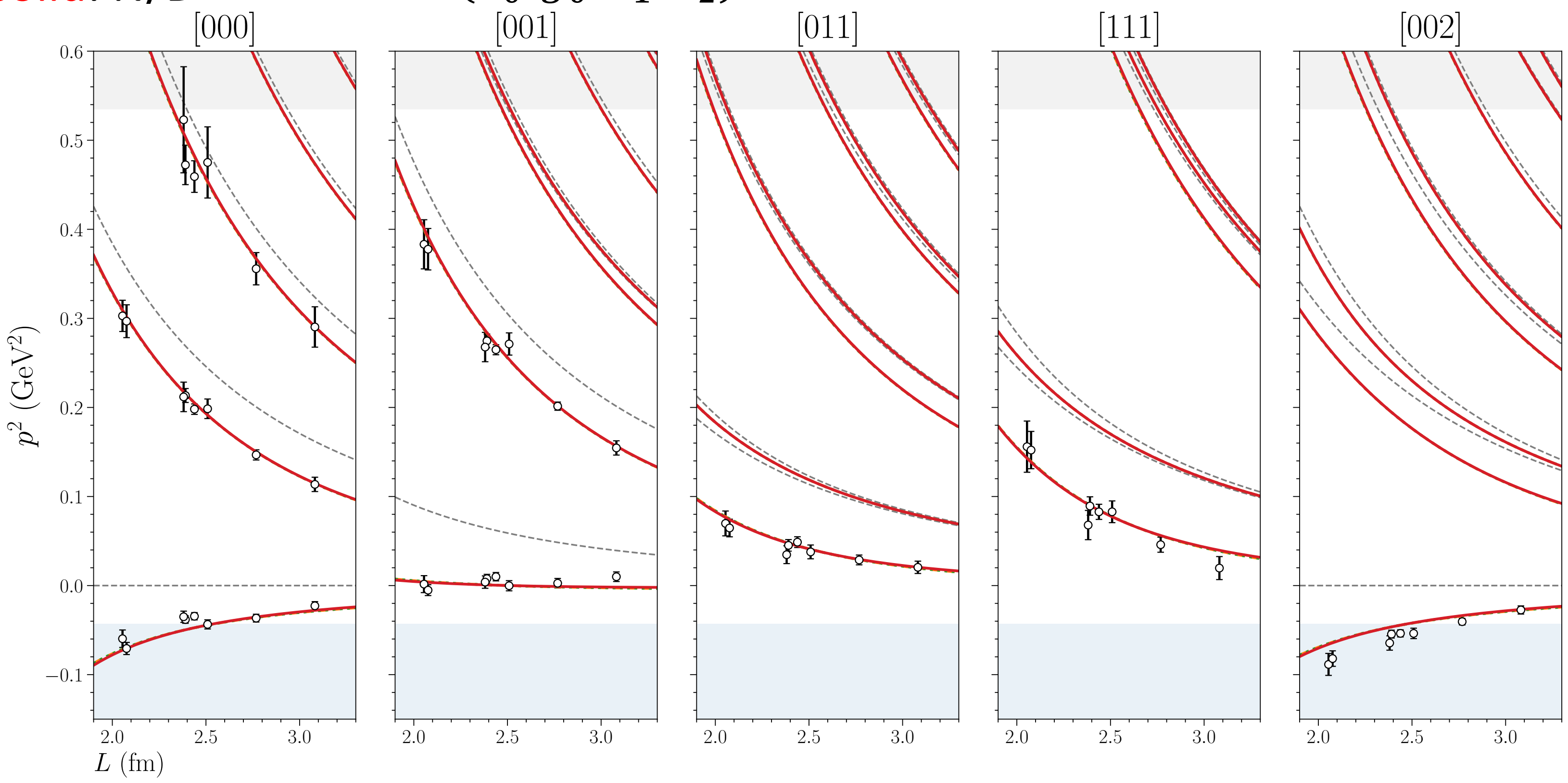
green dotted: ERE CM

red solid: N/D # (n_0, g_0, d_1, d_2)

Thanks to R.J.Perry for numerical support!

From J.R.Green's talk





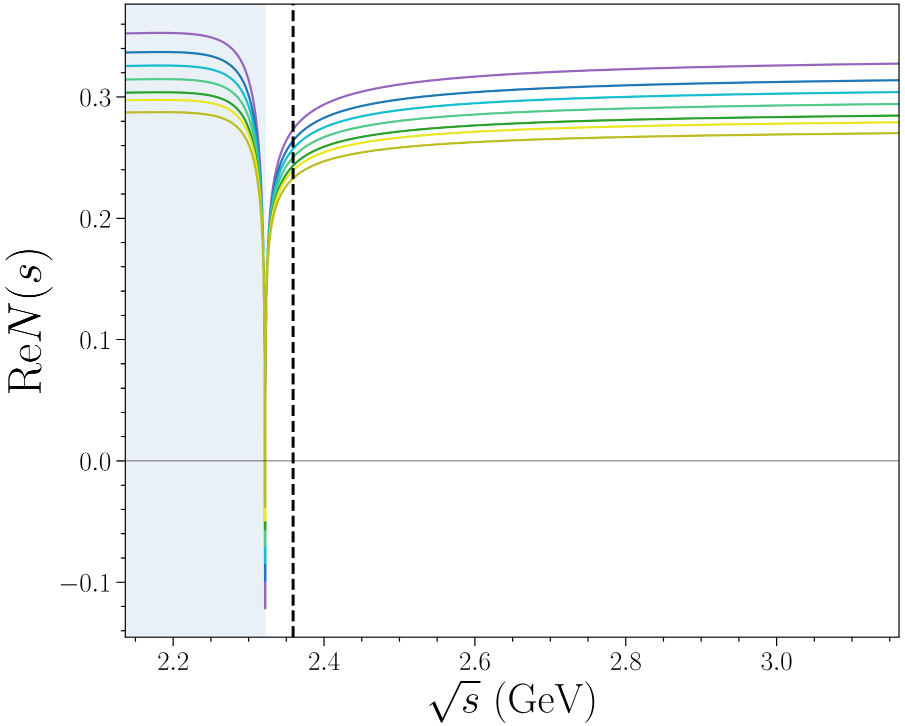
no crossing between (non)interacting levels

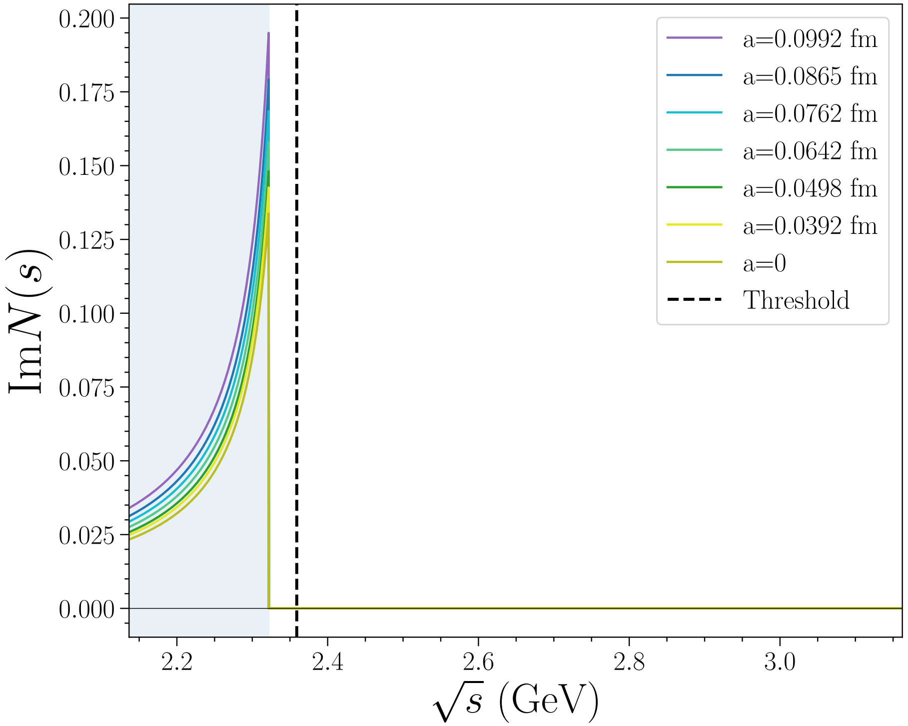
We will see the robustness of N/D method in observables later!

N function and absence of CDD poles

$$t(s) = \frac{N(s)}{D(s)} \quad \rightarrow \quad \frac{\tilde{N}}{\frac{1}{s - s_z} + \cdots}$$

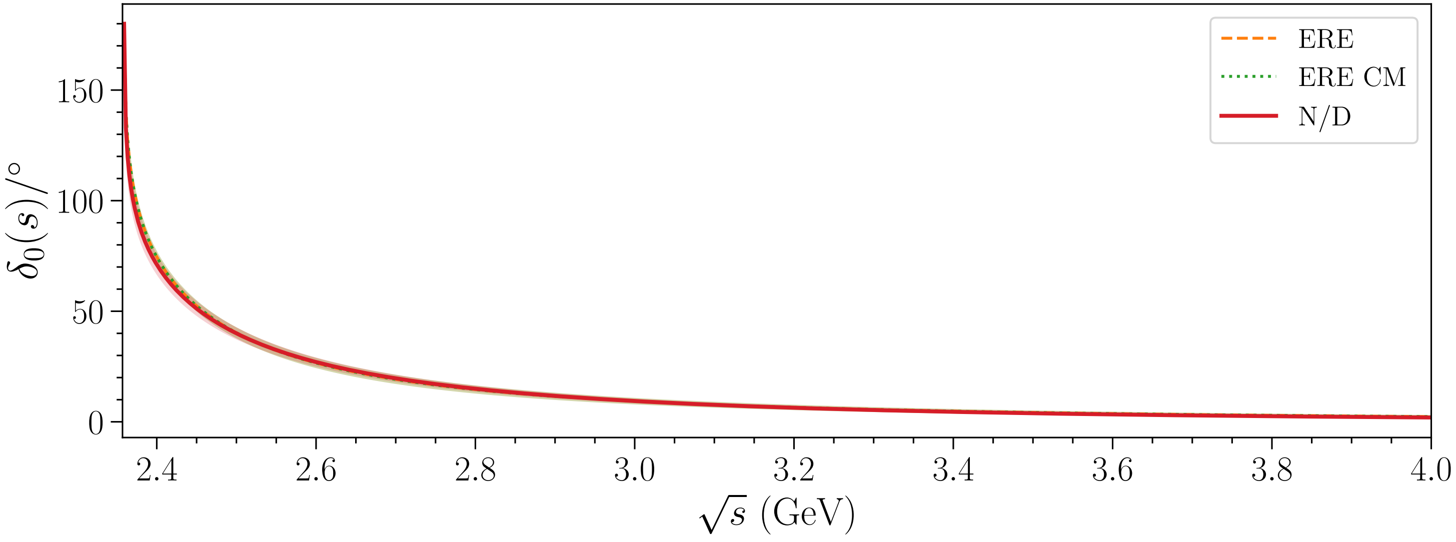
manifests around lhc





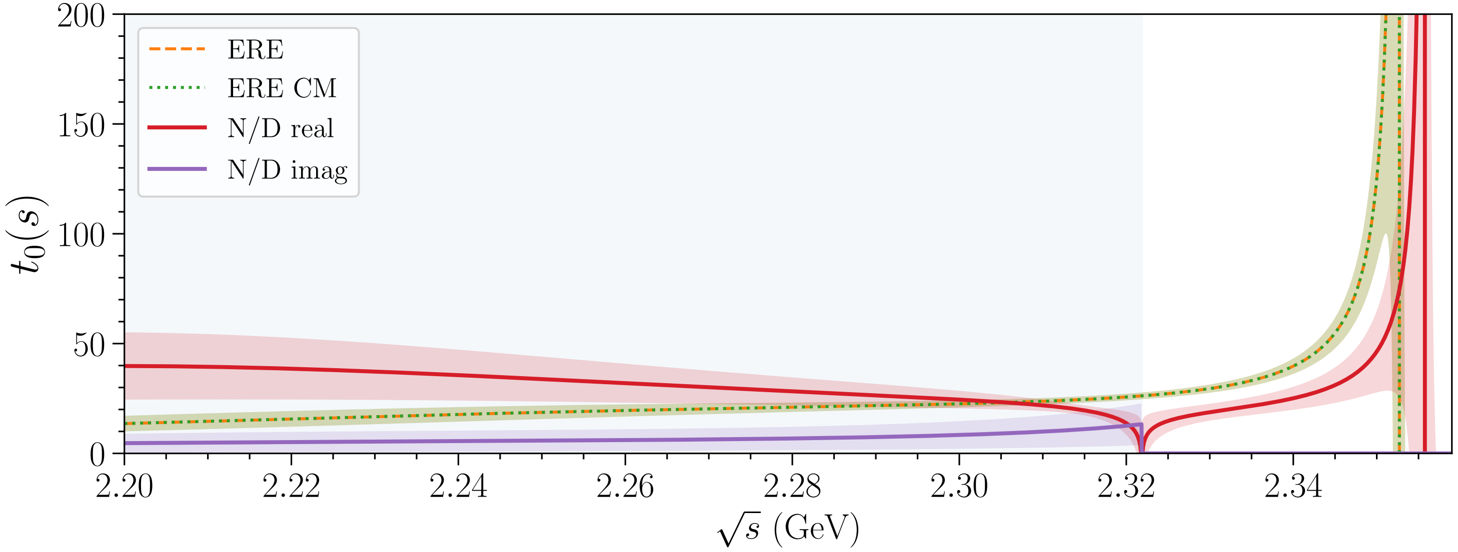
IFV amplitudes

results in the continuum from a-dependent fitting



$s > s_{\text{thr}}$:

- overlapping energy levels ☞ phase shifts
- drastic drop near threshold



$s < s_{\text{thr}}$:

- nonzero imaginary part in N/D
- pole pushed upwards in N/D
- larger uncertainty in N/D

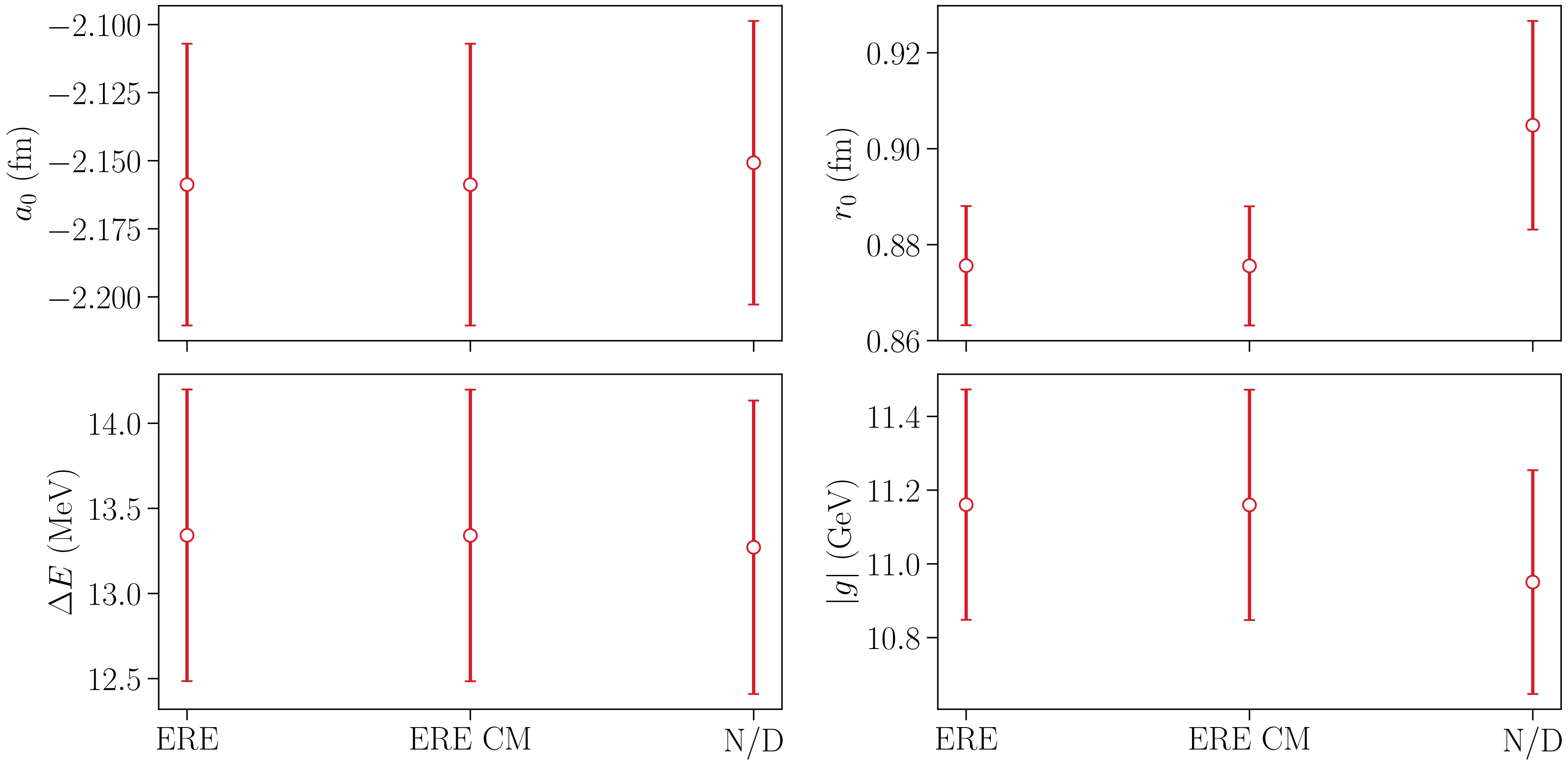
Observables w/o a-dependence

Near threshold:

$$t_0(s) = \frac{\sqrt{s}}{2} \frac{1}{\frac{1}{a_0} + \frac{r_0}{2} p^2 - ip}$$

Near pole:

$$t_0(s) \approx -\frac{1}{16\pi} \frac{|g|^2}{s - s_p}$$



smaller $|a_0|$ and larger r_0 ☞ more bound

Observables w/ a-dependence

Fitting-range:

Ours: [lowest data points, $2m_B + m_\pi$]

Green et al.: [lhc, $2m_B + m_\pi$]

Near threshold:

$$t_0(s) = \frac{\sqrt{s}}{2}\frac{1}{\frac{1}{a_0} + \frac{r_0}{2}p^2 - ip}$$

Near pole:

$$t_0(s) \approx -\frac{1}{16\pi}\frac{|g|^2}{s - s_p}$$

Compositeness:

$$\bar{X}_A = 1 - \bar{Z}_A = \sqrt{\frac{1}{1 + |2r_0/a_0|}}$$

0.778 for deuteron

In the continuum,

systematic uncertainty: 1σ m_B

$a_0 = (-3.737 \pm 0.787 \pm 0.001)\mathrm{fm}$, $r_0 = (0.969 \pm 0.041 \pm 0.001)\mathrm{fm}$

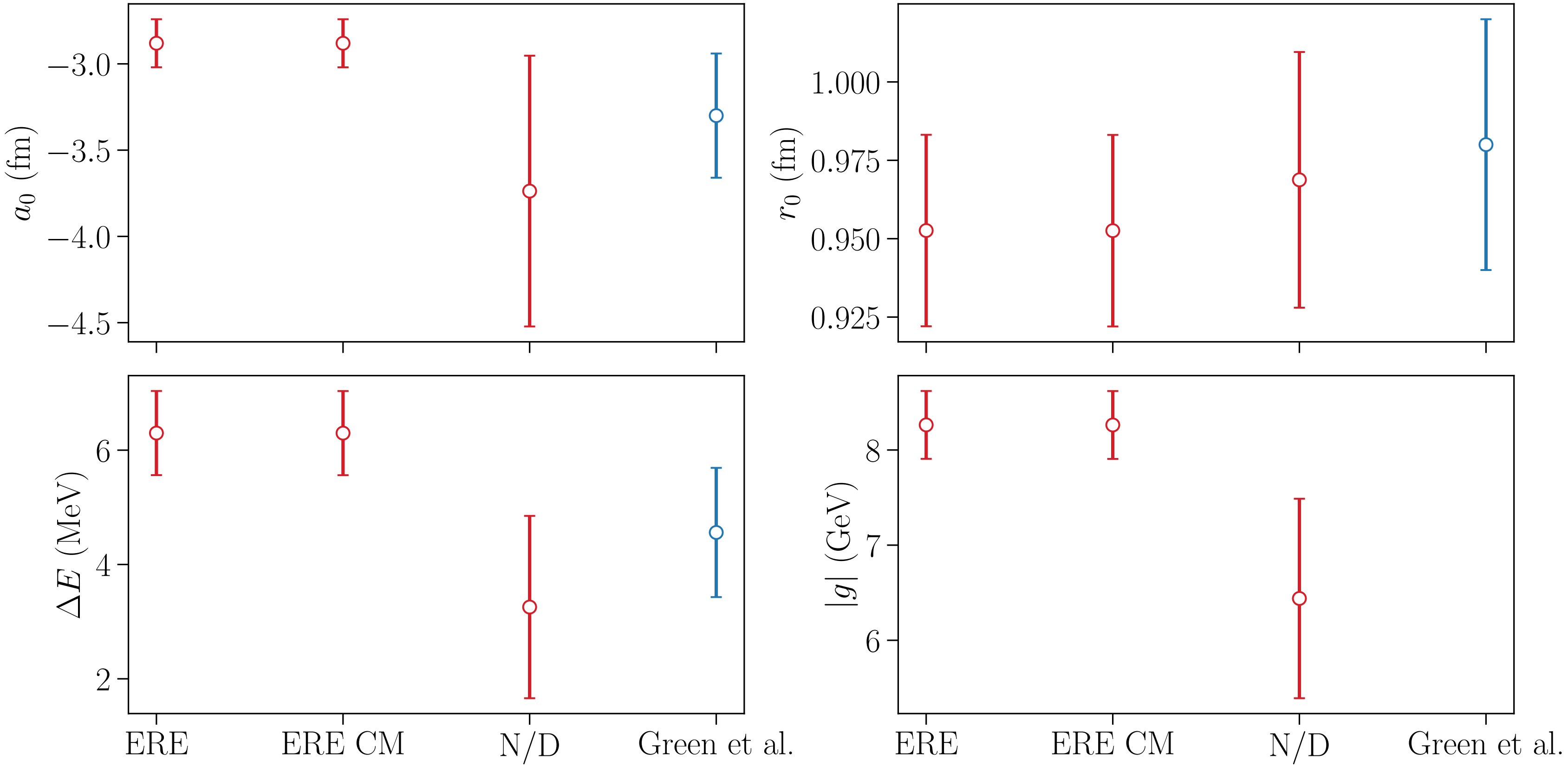
$\Delta E = (3.26 \pm 1.60 \pm 0.04)\mathrm{MeV}$, $|g| = (6.44 \pm 1.05 \pm 0.04)\mathrm{GeV}$

Green et al., PRL127.242003

$\bar{X}_A = 0.8115 \pm 0.0243 \pm 0.0001$

$B_H^{SU(3)_f} = 4.56 \pm 1.13 \pm 0.63\mathrm{MeV}$

0.78 for ERE

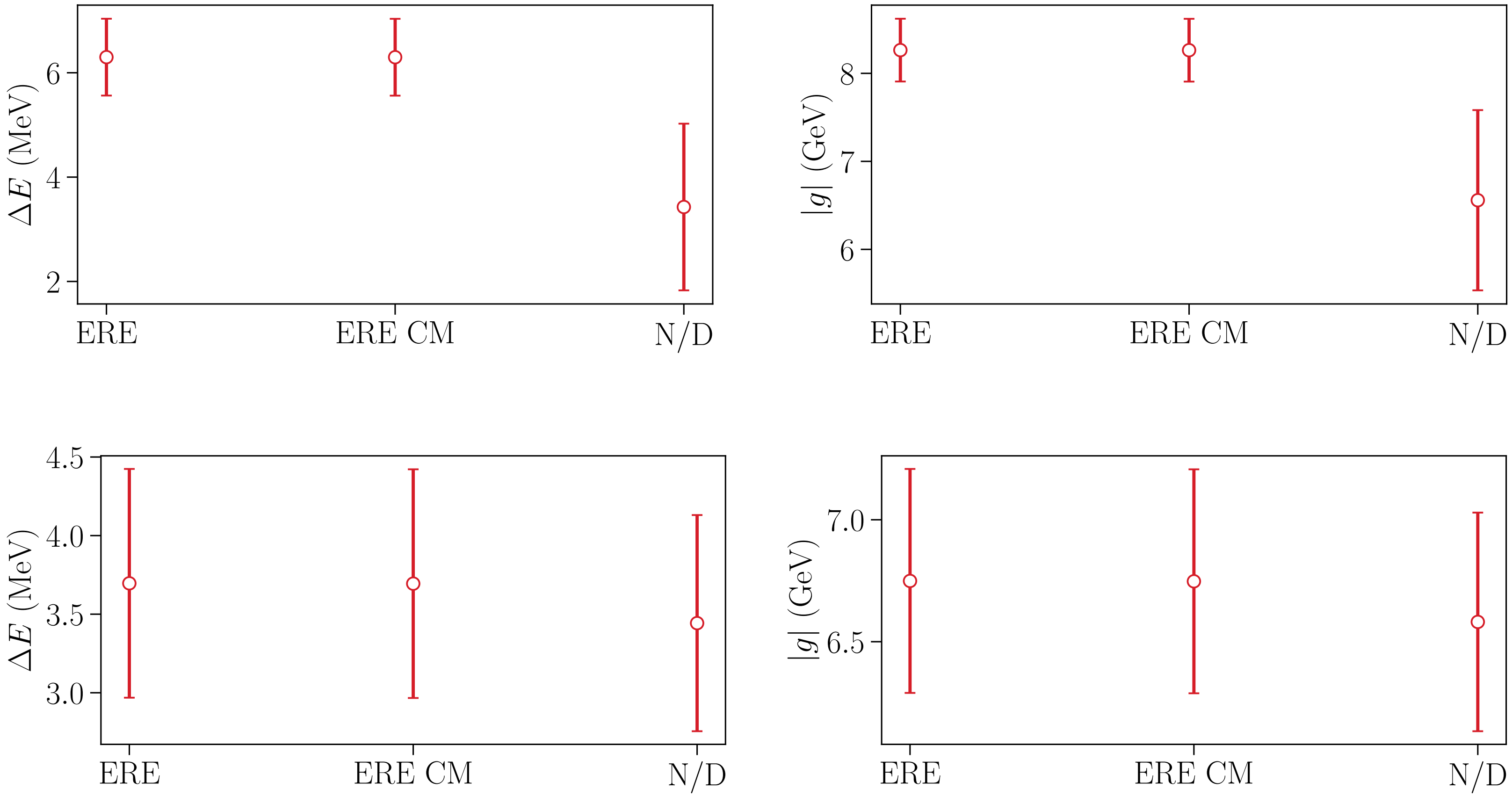


Observables: extrapolation crosscheck

Amplitude extrapolation ☞

$$\begin{tikzcd}
E(L,a) \arrow[r, "\text{lattice quantization?}"] \arrow[d, "a \to 0"'] & \delta(p^2,a) \arrow[d, "a \to 0"] \\
E(L) \arrow[r, "\text{continuum quantization}"'] & \delta(p^2)
\end{tikzcd}$$

Data extrapolation ☞



Our results in N/D are very stable/robust in two different ways of continuum extrapolation while that in ERE change a bit.

Summary

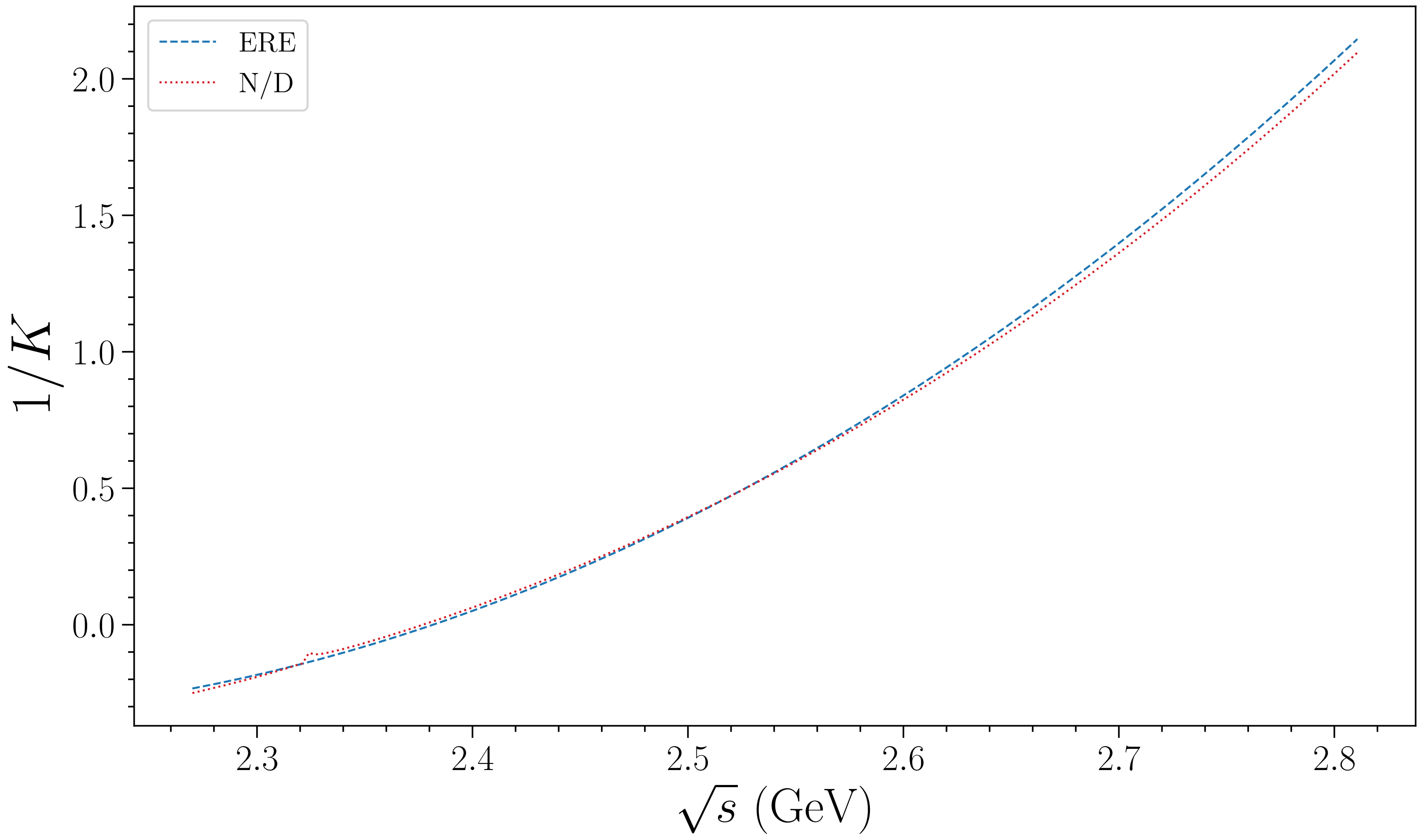
- The left-hand cut is a general challenge for two- and three-hadron scatterings both phenomenologically and on LQCD
- We implemented the first N/D method to FV analysis of H-dibaryon at $\mathrm{SU}(3)_\mathrm{F}$-symmetric point
  - incorporating left-hand-cut effect
  - obtaining reliable results in the continuum
    - ☞ shallow bound H-dibaryon, dominant molecular component
- In the future: $\mathrm{SU}(3)_F$ breaking? Coupled-channel study? Three-body scattering?

  $\Lambda\Lambda - \Sigma\Sigma - \mathrm{N}\Xi$ NN (Deuteron), DD^* etc

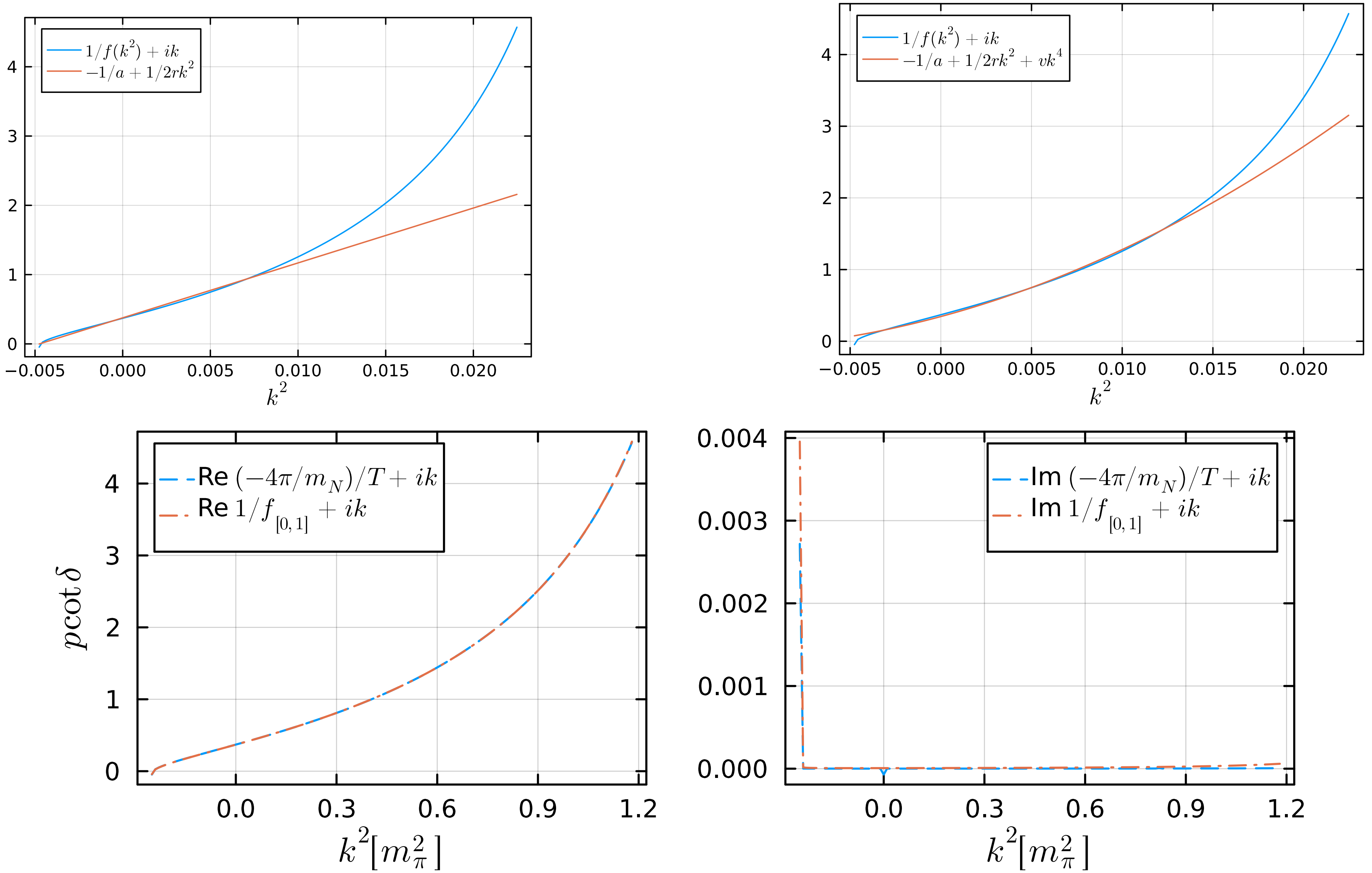
Thank you!

Backup

Inverse of K matrix



Demo of ERE with lhc: virtual state



Glimpse of fitted parameters

ERE w/ a

$$
\begin{aligned}
c_{00} &= -14.60(71) \\
c_{0a} &= 25.3(35) \\
c_{10} &= 2.414(77) \\
c_{1a} &= -1.34(53) \\
c_{20} &= 4.91(80) \\
c_{2a} &= -6.9(48)
\end{aligned}
\qquad
\begin{bmatrix}
1.00 & -0.95 & -0.61 & 0.45 & -0.17 & 0.11 \\
 & 1.00 & 0.65 & -0.58 & 0.16 & -0.13 \\
 & & 1.00 & -0.91 & 0.58 & -0.58 \\
 & & & 1.00 & -0.51 & 0.64 \\
 & & & & 1.00 & -0.91 \\
 & & & & & 1.00
\end{bmatrix}
$$

$$\chi^2/N_{\text{dof}} = \frac{41.91}{61-6} = 0.76.$$

N/D w/ a

$$
\begin{aligned}
n_{00} &= 0.280(25) \\
n_{0a} &= 0.24(18) \\
g_{00} &= 0.089(25) \\
g_{0a} &= 0.07(14) \\
d_{00} &= -0.4173(17) \\
d_{0a} &= 0.0342(93) \\
d_{10} &= 0.04780(43) \\
d_{1a} &= -0.0041(28)
\end{aligned}
\qquad
\begin{bmatrix}
1.00 & -0.85 & -0.03 & -0.03 & 0.35 & -0.49 & 0.73 & -0.61 \\
 & 1.00 & -0.08 & 0.09 & -0.52 & 0.34 & -0.39 & 0.84 \\
 & & 1.00 & -0.77 & -0.35 & 0.17 & -0.12 & 0.05 \\
 & & & 1.00 & 0.10 & -0.34 & 0.15 & 0.03 \\
 & & & & 1.00 & -0.37 & -0.30 & -0.38 \\
 & & & & & 1.00 & -0.26 & -0.16 \\
 & & & & & & 1.00 & -0.32 \\
 & & & & & & & 1.00
\end{bmatrix}
$$

$$\chi^2/N_{\text{dof}} = \frac{40.35}{61-8} = 0.76.$$

ERE w/o a

$$
\begin{aligned}
c_0 &= -10.94(26) \\
c_1 &= 2.219(31) \\
c_2 &= 3.85(33)
\end{aligned}
\qquad
\begin{bmatrix}
1.00 & -0.68 & -0.30 \\
 & 1.00 & 0.67 \\
 & & 1.00
\end{bmatrix}
$$

$$\chi^2/N_{\text{dof}} = \frac{95.83}{61-3} = 1.65.$$

N/D w/o a

$$
\begin{aligned}
n_0 &= 0.317(12) \\
g_0 &= 0.097(17) \\
d_0 &= -0.4127(13) \\
d_1 &= 0.04734(35)
\end{aligned}
\qquad
\begin{bmatrix}
1.00 & -0.40 & -0.18 & 0.83 \\
 & 1.00 & -0.57 & -0.11 \\
 & & 1.00 & -0.67 \\
 & & & 1.00
\end{bmatrix}
$$

$$\chi^2/N_{\text{dof}} = \frac{94.30}{61-4} = 1.65.$$

Amplitude zeros and CDD poles

As explained in the main text, one crucial point of the discussion about the N/D method is the presence or lack of Castillejo-Dalitz-Dyson (CDD) poles [67], i.e., poles of the denominator $D(s)$, which are not explicitly included in our parameterization. However, they can still be identified by zeros of the numerator above threshold. Without loss of generality, we suppose that there is only one simple root s_z above threshold found in the numerator after fitting, i.e., $N(s_z) = 0$. As the amplitude does not change if one divides both $N(s)$ and $D(s)$ by a common factor, we can redefine the numerator and denominator as

$$\tilde{N}(s) = \frac{N(s)}{s - s_z}, \tag{S1}$$

$$\tilde{D}(s) = \frac{D(s)}{s - s_z} = \frac{\tilde{d}_{-1}}{s - s_z} + \sum_{j=0}^{n-1} \tilde{d}_j s^j + \frac{s}{\pi(s - s_z)} \int_{s_{\rm thr}}^{\infty} ds' \frac{\rho(s')N(s')}{s'(s - s')}, \tag{S2}$$

where $\tilde{d}_j$ are polynomials in s_z and the expansion coefficients d_j of $D(s)$, and $\tilde{N}(s)$ is now free of zeros. Using the relation,

$$\frac{N(s')}{s - s_z} = \frac{N(s')}{s' - s_z} + \frac{N(s')(s' - s)}{(s' - s_z)(s - s_z)} = \tilde{N}(s') + \frac{s' - s}{s - s_z}\tilde{N}(s'), \tag{S3}$$

we have

$$\tilde{D}(s) = \frac{\tilde{d}_{-1}}{s - s_z} + \tilde{d}_0 + \sum_{j=1}^{n-1} \tilde{d}_j s^j + \frac{s}{\pi} \int_{s_{\rm thr}}^{\infty} ds' \frac{\rho(s')\tilde{N}(s')}{s'(s - s')} - \frac{s}{\pi(s - s_z)} \int_{s_{\rm thr}}^{\infty} ds' \frac{\rho(s')\tilde{N}(s')}{s'}, \tag{S4}$$

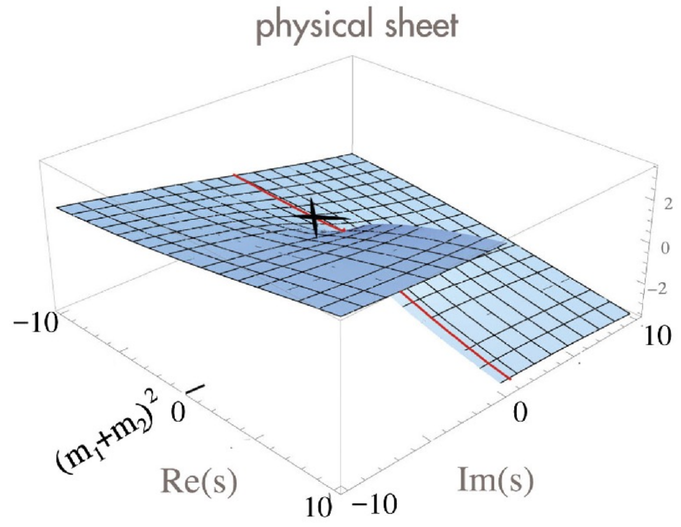
where the second integral is convergent and can be absorbed into the first two terms. Finally, we obtain

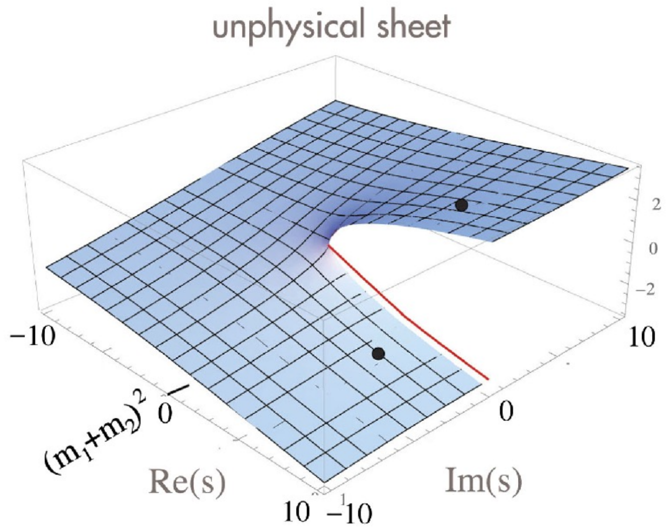
$$\tilde{D}(s) = \frac{\tilde{d}'_{-1}}{s - s_z} + \tilde{d}'_0 + \sum_{j=1}^{n-1} \tilde{d}_j s^j + \frac{s}{\pi} \int_{s_{\rm thr}}^{\infty} ds' \frac{\rho(s')\tilde{N}(s')}{s'(s - s')}, \tag{S5}$$

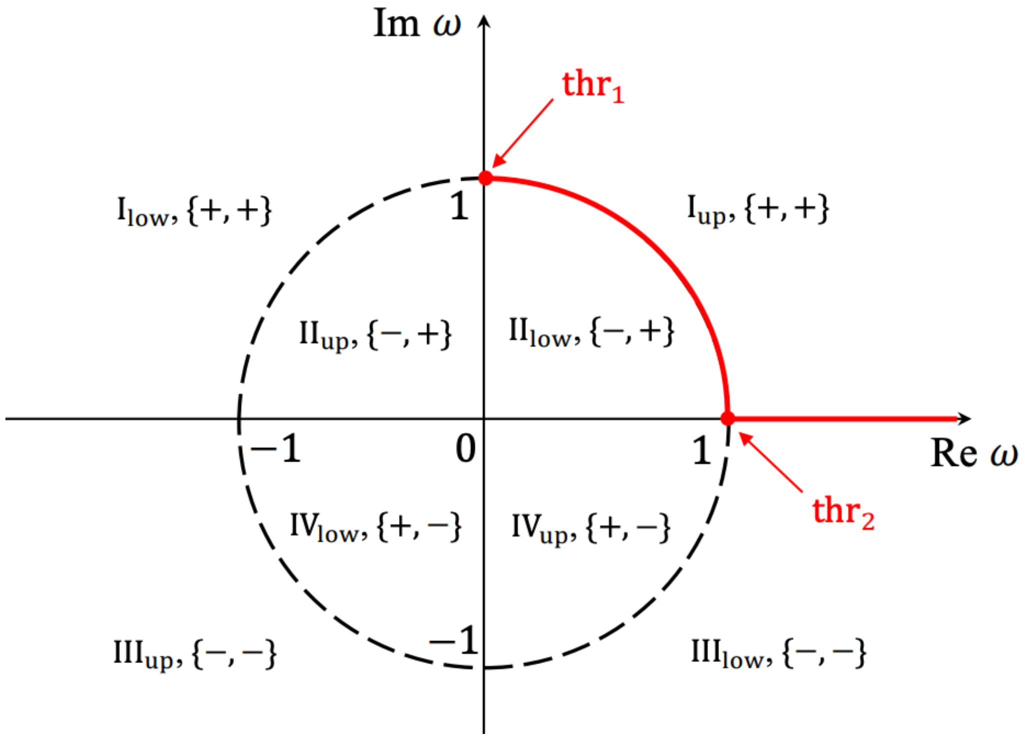
where we can further normalize $\tilde{D}(s)$ by dividing both the new numerator and denominator by $\tilde{d}'_0$, and thus we conclude that the need for additional CDD poles can be accommodated by the zeros of the numerator.

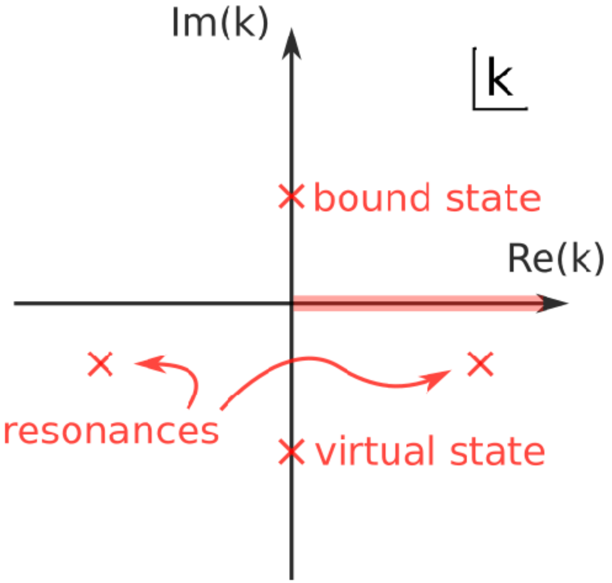
bound,virtual,resonance,coupled-channel...

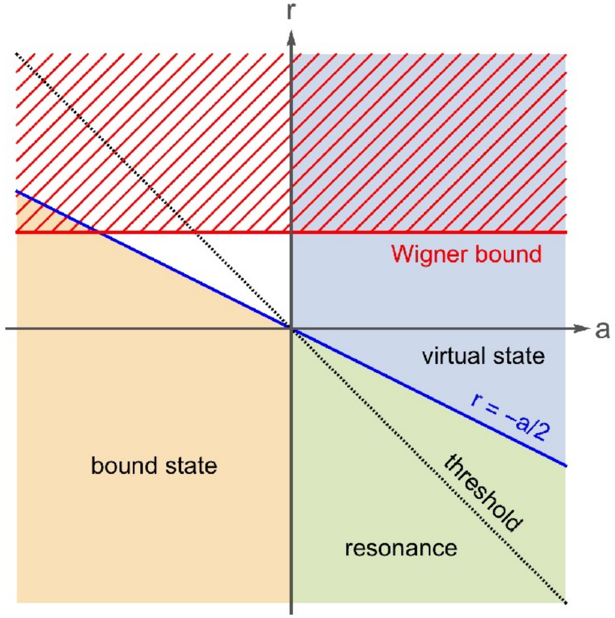
F.K.Guo et al., RevModPhys.90.015004
Inka Matuschek et al., *EPJA*57 (2021) 3, 101
P.Y.Niu et al., *Phys.Rev.D* 110 (2024) 9, 094020











$$\bar{X}_A = 1 - \bar{Z}_A = \sqrt{\frac{1}{1 + |2r_0/a_0|}},$$

Power counting in heavy hadronic molecules

For $HH - HH^* - H^*H^*$ coupling, $\mathcal{Q} \sim m_\pi$, $\Lambda \sim \Lambda_\chi$:

M.P.Valderrama, PRD.85.114037

- Coupled-channel effect is suppressed;

$$\frac{G_\beta(E)}{G_\alpha(E)} = \frac{2\mu_\alpha(E - m_{\alpha1} - m_{\alpha2}) - \vec{q}^2}{2\mu_\beta(E - m_{\beta1} - m_{\beta2} - \Delta_{\alpha\beta}) - \vec{q}^2} = (\frac{Q}{\Lambda_{CC}})^2, \quad \Lambda_{CC} = \sqrt{2\mu_{HH}\Delta_Q}$$

  - $\Lambda_{CC} \gtrsim 500$ MeV for charm&bottom sectors;
  - $m_Q \to \infty, \Lambda_C \sim m_Q^0 \quad \Rightarrow \quad$ HQSS;

The OPE can be treated perturbatively for charmed hadronic molecule!

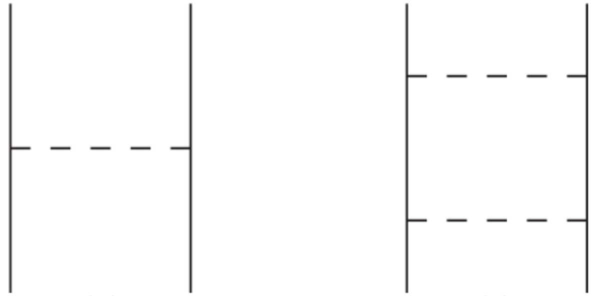
- The pion-exchange can be treated perturbatively mostly:

$$V(q) = V^{Contact}(q) + V^{OPE}(q)$$

  - If $V^{Contact} \sim \mathcal{Q}^{-1}$;
  - Central part of OPE:

$$\frac{\langle p' | V^{OPE} G V^{OPE} | p \rangle}{\langle p' | V^{OPE} | p \rangle} \sim \frac{\mathcal{Q}}{\Lambda_C}, \quad \Lambda_C = \frac{1}{|\sigma\tau|}\frac{24\pi f_\pi^2}{\mu_{HH} g^2} \gtrsim 0.6\text{GeV}(I = 0,\ S = 0\ D^*D^*)$$

  - Tensor part of OPE is valid up to $E_B \sim$ decades of MeV



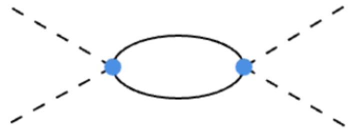
(a) (b)

What if we promote OPE into LO by reserving the δ-term incautiously?

Non-perturbative treatment on OPE

◉At loop level,

- Hard-cutoff or FFs will spoil the above power counting;
- Dimensional regularization gives $1/\epsilon$.



$$i\mathscr{M} \propto (\frac{p^2}{\Lambda_\chi^2})^2 i\int \frac{d^4q}{(2\pi)^4}\frac{1}{((P-q)^2-m_1^2)(q^2-m_2^2)}$$

◉Interactions @1-loop,

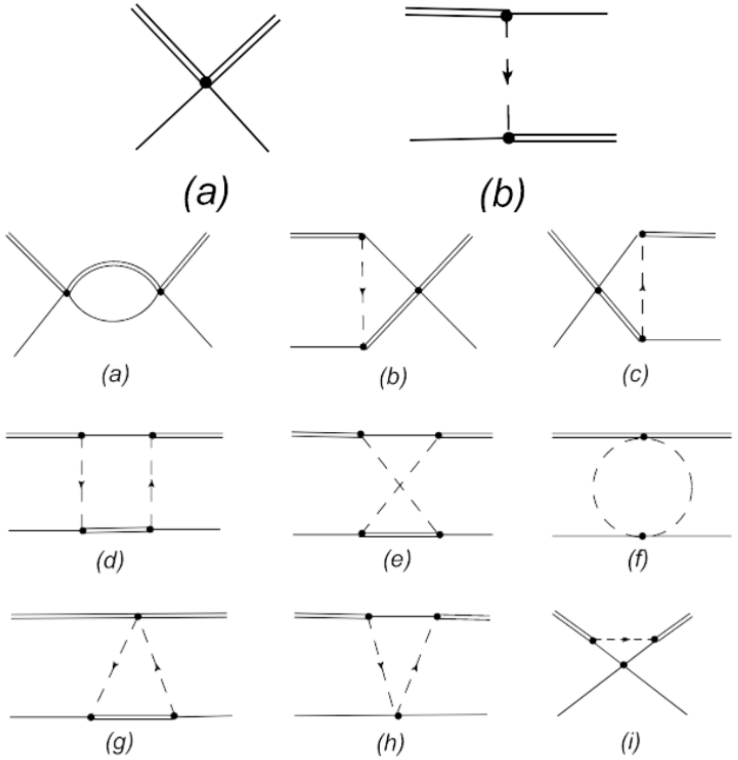
P.Wang et al., PRL.111.042002
V.Baru et al.,PRD.91.034002

- DR

$$\mathscr{A}_{SS}^{(1),a} = i\lambda^2\frac{p}{8\pi(M_D+M_{D^*})}$$

- Hard-cutoff

$$\mathscr{A}_{SS}^{(1),a} = \frac{\lambda^2}{8\pi(M_D+M_{D^*})}(\frac{2}{\pi}\Lambda + ip)$$



◉It requires counter term,

$$\delta\mathscr{A}_{SS}^{(1)} = -\frac{\Lambda}{\pi^2(M_D+M_{D^*})}(\frac{\lambda^2}{4}+\lambda\frac{2g_\pi^2}{3F^2}+\frac{4g_\pi^4}{3F^4})$$

η, ρ, ω, etc $\qquad \pi$

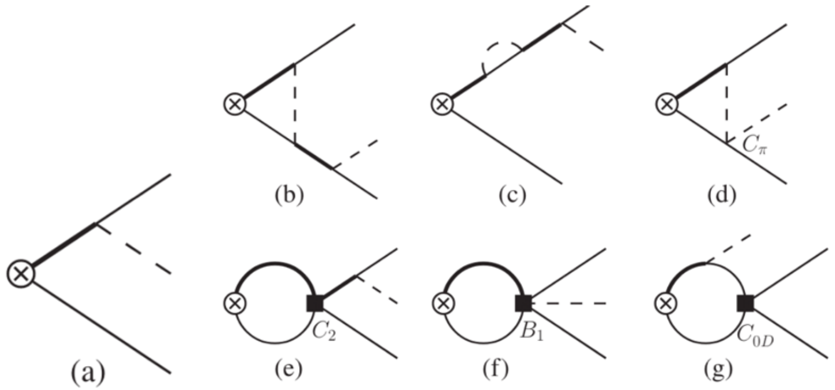
◉Schemes:

- Contact potential + OPE;
- How about OBE?

EFT's perspective & Resonance saturation

S.Fleming et al.,PRD76.034006

$$\mathcal{Q} \sim m_\pi \ll \Lambda_0 \sim m_\rho \ll M_{J/\psi}$$
$$R \sim 1.5F \gg \sim 0.25F \gg < 0.1F$$

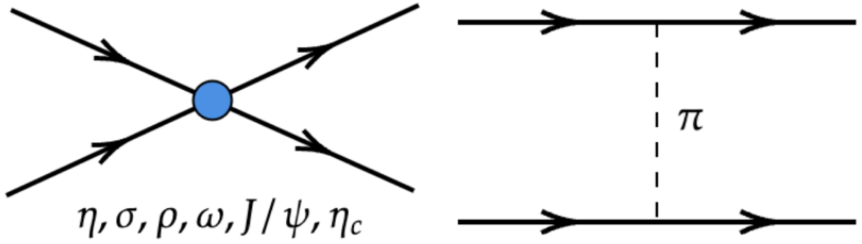


Lin Dai et al., PRD.101.054024
M.P.Valderrama, PRD.85.114037

Naive dimensional analysis:

- LO contact potential ⇔ VMD
- Perturbative pionful interaction(NLO)
- Coupled-channel effect suppressed(NLO)

F.Z.Peng et al., PRD.102.114020



- center terms:

$$V(\boldsymbol{p}, \boldsymbol{p'}) \propto \frac{1}{\boldsymbol{q}^2 + \mu^2}$$

- tensor terms:

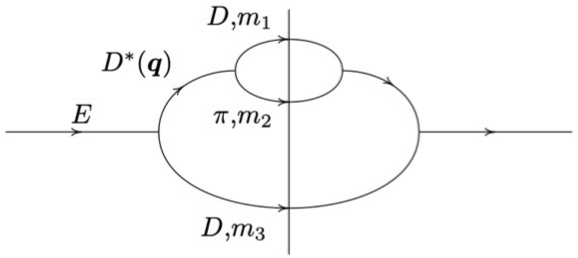
$$V(\boldsymbol{p}, \boldsymbol{p'}) \propto \frac{\boldsymbol{\epsilon_i} \cdot \boldsymbol{q} \boldsymbol{\epsilon_f} \cdot \boldsymbol{q}}{\boldsymbol{q}^2 + \mu^2}$$

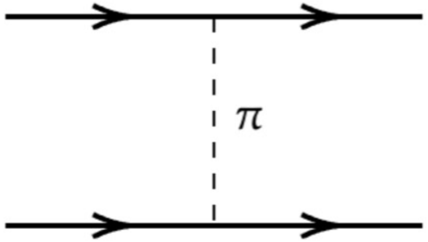
$$\propto \frac{1}{3}\left(1 - \frac{\mu^2}{\boldsymbol{q}^2 + \mu^2}\right)\hat{S}(\boldsymbol{\epsilon_i}, \boldsymbol{\epsilon_f}) + \frac{\boldsymbol{q}^2}{\boldsymbol{q}^2 + \mu^2}\hat{T}(\boldsymbol{\epsilon_i}, \boldsymbol{\epsilon_f})$$

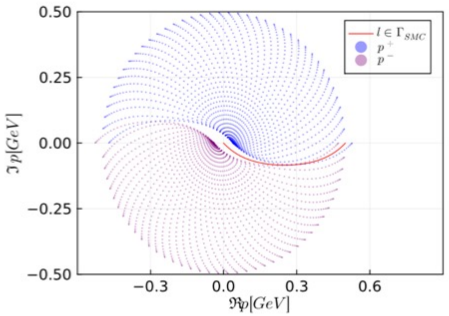
N.Yalikun et al., PhysRevD.104.094039 T.Ji et al., PRL.129.102002

introduce counter terms: $1 \to C_{P/V}(\Lambda)$

Analytical continuation







1) $D^*[D\pi]$ self energy

$$p_{cm} = \mathcal{F}(E, \boldsymbol{q}, m_1, m_2, m_3) = \sqrt{2\mu_{D\pi}(E + m_{D^*} - \sqrt{m_{D^*}^2 + \boldsymbol{q}^2} - \sqrt{m_D^2 + \boldsymbol{q}^2} - m_D - m_\pi)}$$

$$\tilde{\mathcal{F}}(E, \boldsymbol{q}, m_1, m_2, m_3) = \begin{cases} -\mathcal{F}(E, \boldsymbol{q}, m_1, m_2, m_3), & \text{if } \Im(E'(E, \boldsymbol{q})) < 0 \text{ and } \Re(E'(E, \boldsymbol{q})) > m_1 + m_2 \\ \mathcal{F}(E, \boldsymbol{q}, m_1, m_2, m_3), & \text{else} \end{cases}$$

2) OPE:

$$\int_{-1}^{1} \frac{dz}{z-\xi} = \begin{cases} \log(1-\xi) - \log(-1-\xi) + 2\pi i, & |\Re\xi| < 1 \text{ and } \Im\xi < 0 \\ \log(1-\xi) - \log(-1-\xi), & \text{else} \end{cases}$$

with $\xi = \frac{p_1^2 + p_4^2 + m_\pi^2 - (E_4 - E_1)^2}{2 p_1 p_4}$.